The Stone-von Neumann Construction in Branching Rules and Minimal Degree Problems

Camelia Karimianpour

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Department of Mathematics and Statistics
Faculty of Science
University of Ottawa

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Abstract

In Part I, we investigate the principal series representations of the n-fold covering groups of the special linear group over a p-adic field. Such representations are constructed via the Stone-von Neumann theorem. We have three interrelated results. We first compute the K-types of these representations. We then give a complete set of reducibility points for the unramified principal series representations. Among these are the unitary unramified principal series representations, for which we further investigate the distribution of the K-types among its irreducible components.

In Part II, we demonstrate another application of the Stone-von Neumann theorem. Namely, we present a lower bound for the minimal degree of a faithful representation of an adjoint Chevalley group over a quotient ring of a non-Archimedean local field.

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Dedication

to my parents

Contents

List of Symbols

$P_{\mathbf{s}}$	$P_{\mathbf{s}} : C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}) \to \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$	96
φ_0	quasi-characteristic function of $\widetilde{K}$	97
α	$\alpha : \pi_{\mathbf{s}} \to \rho_{\mathbf{s}}, f \mapsto f(1)$	98
$\mathrm{In}_{\widetilde{w}}$	$\ker(\alpha)$	99
$\pi_{\widetilde{w}}$	$\pi_{\mathbf{s}}\|_{\widetilde{B}}$	99
$\Lambda_{\widetilde{w},\mathbf{s}}$	$\Lambda_{\widetilde{w},\mathbf{s}} : \mathrm{In}_{\widetilde{w}} \to \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2}$	101
$\Lambda_{\mathbf{s}}$	extension of $\Lambda_{\widetilde{w},\mathbf{s}}$ to $\pi_{\mathbf{s}}$	105
$\mathcal{T}_{\mathbf{s}}$	$\mathcal{T}_{\mathbf{s}} : \pi_{\mathbf{s}} \to \pi_{-\mathbf{s}+2}$	105
$\mathcal{T}$	$\mathcal{T}_{-\mathbf{s}+2} \circ \mathcal{T}_{\mathbf{s}}$	106
$D_{\mathbf{s}}$	$\Lambda_{\mathbf{s}} \circ P_{\mathbf{s}}$	110
N^+	$N \cap \widetilde{K}$	117
N^-	$\{(\mathrm{lt}(\varpi x), 1) \mid x \in \mathcal{O}\}$	117
$\widetilde{I}_0$	$\widetilde{I} \cap \widetilde{K}_0$	117
$\widetilde{T}_0$	$\widetilde{T} \cap \widetilde{K}_0$	117
$\boldsymbol{\eta}$	$\boldsymbol{\eta} : \mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho \to L^2(\mathbb{F}, \mathbb{C}^n)$	130
$m_{\mathfrak{f}}(\cdot)$	minimal degree of a faithful representation of a group	164
$\mathfrak{g}$	complex finite-dimensional simple Lie algebra	171
$\mathfrak{h}$	Cartan subalgebra of $\mathfrak{g}$	171
$\mathfrak{q}$	Heisenberg parabolic subalgebra	172
$\mathbf{G}_{ad}$	elementary adjoint Chevalley group	176

Introduction

For us, covering groups, also known in the literature as metaplectic groups, are central extensions of a simply-connected simple and split algebraic group over a local field $\mathbb{F}$ by the group of the n-th roots of unity, μ_n. Such groups fall into the category of locally profinite groups. In the case $n = 2$, such covering groups were first presented explicitly in Weil's memoir [Wei64] for the symplectic group. The problem of determining this class of groups, in the more general case, was studied by Steinberg [Ste68] and Moore [Moo68] in 1968, and further completed by Matsumoto [Mat69] in 1969 for simply-connected Chevalley groups. In these works, one considers Steinberg's explicit construction, by generators and relations, of the covering groups. Around the same time, Kubota independently constructed n-fold covering groups of SL_2 [Kub67] and GL_2 [Kub69] over a p-adic field by means of presenting an explicit 2-cocycle. Kubota's cocycle is expressed in terms of the n-th Hilbert symbol.

Since then, there have been a number of studies of representations of this class of groups from different perspectives, among them being the work of H. Aritürk [Ari80], D. A. Kazhdan and S. J. Patterson [KP84], C. Moen [Moe88], D. Joyner [Joy98, Joy01], G. Savin [Sav04], M. Weissman and T. Howard [HW09], and P. J. McNamara [McN12].

In the first part of this thesis, my objective is to study principal series representations of the n-fold covering groups $\widetilde{\mathrm{SL}}_2$ of SL_2 over a p-adic field $\mathbb{F}$ for $n > 2$, introduced by McNamara [McN12]. Since this thesis is intended to be expository and self-contained, we draw on some of the work listed above (with specific citations). Our approach is to work with the explicit construction of $\widetilde{\mathrm{SL}}_2$ given by Kubota [Kub67]. We assume $n|q-1$, where q is the size of the residue field of the p-adic field. This implies the splitting of the central extension over a maximal compact subgroup K of $\mathrm{SL}_2(\mathbb{F})$.

The principal series representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ are those representations that are induced from the inverse image $\widetilde{B}$ of a Borel subgroup B of $\mathrm{SL}_2(\mathbb{F})$. The construction of those representations of $\widetilde{B}$ that are trivial on the unipotent radical of $\widetilde{B}$ brings us to the study of the irreducible representations of the metaplectic torus $\widetilde{T}$, i.e., the inverse image of the split torus T of $\mathrm{SL}_2(\mathbb{F})$ in $\widetilde{\mathrm{SL}}_2(\mathbb{F})$.

An important feature of $\widetilde{T}$, which differentiates the nature of its representations from those of a linear torus, is that it is not abelian. However, it is a Heisenberg group and its irreducible representations are governed by the Stone-von Neumann theorem. The Stone-von Neumann theorem characterizes irreducible representations of two-step nilpotent groups, also known as Heisenberg groups, according to their central characters. Indeed, given a character of the centre of a Heisenberg group that satisfies some mild conditions, the Stone-von Neumann theorem provides a recipe to construct the corresponding, unique up to isomorphism, irreducible representation of the Heisenberg group. The construction involves induction from a maximal abelian subgroup of the Heisenberg group.

Once an irreducible representation ρ_χ, with central character χ, of $\widetilde{T}$ is obtained, the principal series representation π of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ is $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}}_2(\mathbb{F})}\rho_\chi$, where ρ_χ is trivially extended on the unipotent radical subgroup of $\widetilde{B}$. These representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ were introduced in [McN12], and they admit several open questions. We take the following two approaches in order to study π_χ. On the one hand, we investigate the K-types of π_χ. On the other hand, we determine the unramified characters χ for which π_χ is reducible. Finally, the two trajectories meet when we investigate the distribution of the K-types among the irreducible constituents of π_χ. Let us now elaborate on each of these three streams.

A common technique in representation theory is to study the decomposition of the restriction of representations to a particular subgroup. In the theory of real Lie groups, restriction to maximal compact subgroups retains a lot of information from the representation; in fact, such a restriction is a key step towards classifying irreducible unitary representations. In the case of reductive groups over p-adic fields, investigating the decomposition upon restriction to maximal compact subgroups reveals a finer structure of the representation, in the interests of recovering essential information about the original representation.

The problem of decomposing a representation of a reductive p-adic group upon restriction to a maximal compact, which we refer to as the K-type problem, is visited and solved in certain cases, including the principal series representations of GL(3) [CN09, CN10, OS14], and SL(2) [Nev05, Nev11], representations of GL(2) [Cas73], and supercuspidal representations of SL(2) [Nev13].

Our result is as follows. Let $\mathcal{O}$ be the ring of integers of $\mathbb{F}$ and let $\widetilde{K}$ be the inverse image of the maximal compact subgroup $\mathrm{SL}_2(\mathcal{O})$ in $\widetilde{\mathrm{SL}_2}(\mathbb{F})$. The representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}_2}(\mathbb{F})}\rho_\chi$ is smooth; it can be viewed as the union of its K_i-fixed points, where the K_i's are the natural filtration of $\widetilde{K}$. This allows us to reduce the problem to calculating the dimensions of certain finite-dimensional Hecke algebras. The key calculation for determining the decomposition is the determination of certain double cosets that support intertwining operators for the restricted principal series representation. Our main result, which is stated in Theorem 4.4.19, can be summarized as follows:

Theorem 1. *Let χ be a character of $Z(\widetilde{T})$ of depth $m-1$ and let $\underline{n} = n$ if n is odd, and $\underline{n} = n/2$ if n is even. Then $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}_2}(\mathbb{F})}\rho_\chi$ decomposes into a direct sum of $\widetilde{K}$-representations as:*

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}_2}(\mathbb{F})}\rho_\chi \simeq \bigoplus_{i=0}^{\underline{n}-1}\left((\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_m}\right) \oplus \bigoplus_{l>m}\left(\widetilde{W_{0,l}^{+}} \oplus \widetilde{W_{0,l}^{-}}\right)^{\oplus\underline{n}},$$

where the χ_i*'s are all possible extensions of* χ *to a maximal abelian subgroup* A *of* $\widetilde{T}$*, and* $\widetilde{W}_{0,l}^{+}$ *and* $\widetilde{W}_{0,l}^{-}$ *are two irreducible inequivalent spaces of the same degree such that* $\widetilde{W}_{0,l}^{+} \oplus \widetilde{W}_{0,l}^{-} \cong (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi_0)^{K_l} / (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi_0)^{K_{l-1}}$*. Moreover, under the condition* $m > 1$ *all the summands are irreducible.*

A key step in understanding the degrees of $\widetilde{W}_{0,l}^{-}$ and $\widetilde{W}_{0,l}^{+}$ in the decomposition is to investigate the K-type problem for the n-fold cover of $\mathrm{GL}_2(\mathbb{F})$ defined in [Kub69]. The final result is obtained by comparing the restriction to $\widetilde{K}$ of the principal series representation of $\widetilde{\mathrm{GL}}_2(\mathbb{F})$ with $\mathrm{Res}_{\widetilde{K}} \mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{SL}}_2(\mathbb{F})} \rho_\chi$. In my work, I follow the argument in [Nev05] for the linear group $\mathrm{SL}_2(\mathbb{F})$; however, the technicalities in the covering case are much more involved than the linear case, and the results are fairly different. For instance, the decomposition is no longer multiplicity-free (see Corollary 4.2.15).

Having established the K-types, we explore π_χ from a different angle; namely, we study the problem of determining the unramified characters χ for which π_χ is reducible. This problem is solved for unitary unramified principal series of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ by Moen in [Moe88]. For non-unitary unramified principal series, the case of $n = 2$ is solved by Gelbart and Sally [GS75], and $n = 3$ is solved by Aritürk [Ari80]. Along the same line, the decomposition of the unramified regular principal series of the n-fold covering groups of certain Chevalley groups is also of interest in [Sav04]. However, he excludes the case of $\mathrm{SL}_2(\mathbb{F})$ from his computation (and explicit statement of results).

In Chapter 5 of this thesis, we undertake the determination of those unramified characters χ for which π_χ is irreducible. As suggested by the study of K-types of π_χ, the behaviour of π_χ depends on the parity of n. We chose to restrict ourselves to odd n and leave the even case for future work. In particular, the result in [Ari80] is a special case of ours. Unramified characters, and hence their corresponding principal series representations, are parameterized by a complex power s of the norm character $|\cdot| : \mathbb{F}^\times \to \mathbb{C}^\times$. Our result, which is stated in Theorems 5.2.13 and 5.3.6, can be summarized as follows:

Theorem 2. *Given an unramified character $\chi_{\mathbf{s}}$ of $Z(\widetilde{T})$, the principal series representation $\pi_{\mathbf{s}}$ is reducible if and only if $\mathbf{s} = 1 \pm \frac{1}{n}$ or $\mathbf{s} = 1 + \frac{\pi i}{\log q}$. For $\mathbf{s} = 1 + \frac{\pi i}{\log q}$, the principal series representation $\pi_{\mathbf{s}}$ decomposes as a direct sum of two inequivalent subrepresentations.*

The proof is adapted from the classical method of finding reducibility points of the principal series representations of the linear group $\mathrm{SL}_2(\mathbb{F})$ [Cas95]. The splitting of the central extension over K implies the existence of an open compact section $\widetilde{K}_0 \subset \widetilde{K}$ isomorphic to K. We deal with regular and non-regular characters $\chi_{\mathbf{s}}$ separately. Let $\phi_{\mathbf{s}}$ be a fixed spherical function in $\pi_{\mathbf{s}}$. In the case of regular characters, the idea is to construct a special intertwining operator $\mathcal{T}$ in $\mathrm{Hom}_{\widetilde{\mathrm{SL}}_2(\mathbb{F})}(\pi_{\mathbf{s}}, \pi_{\mathbf{s}})$ and pin down the image of the spherical function under this map. The crucial tool is an adaptation of [Cas95, Theorem 6.6.2] to the covering group in Proposition 5.2.6. This theorem links the question of the irreducibility of $\pi_{\mathbf{s}}$ to that of the vanishing of $\mathcal{T}$. For a non-regular character $\chi_{\mathbf{s}}$, we prove a variant of Jacquet's first lemma, adapted for the non-linear group $\widetilde{\mathrm{SL}}_2(\mathbb{F})$, which allows us to reduce the problem to the Jacquet module level.

We finish Part I by weaving Theorem 1 and Theorem 2 together in Chapter 6. This is accomplished by investigating the problem of the distribution of the $\widetilde{K}$-irreducible spaces in the K-type decomposition in Theorem 1, into the $\widetilde{\mathrm{SL}}_2(\mathbb{F})$-irreducible constituents $\pi_{\mathbf{s}}$ in Theorem 2, when $\pi_{\mathbf{s}}$ is the reducible unramified unitary representation. My approach, inspired by the argument in [GGPS90] for the linear group $\mathrm{SL}_2(\mathbb{F})$, is to calculate a non-invertible intertwining operator A from the L^2-realization of $\pi_{\mathbf{s}}$ to itself. This involves a careful study of vector valued Bessel and Gamma functions. In fact, calculating the effect of the intertwining operator A on the value of functions at a single point provides a substantial amount of information. In particular, one can identify the irreducible component that a specific K-type belongs to. Hence, to understand the K-type distribution, one should first concretely realize the $\widetilde{K}$-spaces in Theorem 1 as spaces of matrix-valued square integrable functions.

Using the method explained above, I succeeded in verifying that the spherical K-types and its complement, appearing in the first piece of the decomposition in Theorem 1 for $\pi_{\mathbf{s}}$ for $\mathbf{s} = 1 + \frac{\pi i}{\log q}$, belong to opposite components. This result is stated and proved in Theorem 6.2.3 and Theorem 6.2.6.

The second part of my thesis is on the *minimal degree problem*. Let G be a finite group and let $m_{\mathfrak{f}}(G)$ be the smallest possible dimension of a faithful representation of G. The minimal degree problem is to find a "good" lower bound for $m_{\mathfrak{f}}(G)$.

This problem was considered for PSL_2 over a finite field of size p, with p prime, by Frobenius [Fro]. The lower bound given by Frobenius was shown to be serviceable in a number of combinatoric and number theoretic applications [SX91, BG08b]. This bound has been generalized by V. Landazuri and G. M. Seitz and A. E. Zelesski to finite simple groups of Lie type [LS74, SZ93]. Bourgain and Gamburd considered the problem for $\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})$ as a step towards showing that a certain family of graphs is an expander family. Inspired by combinatorial applications of such bounds, M. Bardestani and K. Mallahi-Karai studied the minimal degree problem for $\mathrm{SL}_k(\mathbb{Z}/p^n\mathbb{Z})$ and $\mathrm{Sp}_{2k}(\mathbb{Z}/p^n\mathbb{Z})$ [BMK15].

In Chapter 7, in a joint work with M. Bardestani, K. Mallahi-Karai and H. Salmasian, we study the minimal degree problem for the adjoint Chevalley groups G over $\mathcal{O}/\mathfrak{p}$, where $\mathcal{O}$ is the ring of integers, and $\mathfrak{p}$ is the maximal prime ideal of a non-Archimedean local field F.

The key step in our approach is that we construct a Heisenberg subgroup H of G. To give a lower bound to $m_{\mathfrak{f}}(G)$, given a faithful representation (ρ, V) of G, we show that there exists at least one irreducible constituent (σ_1, V_1) in the decomposition of $\rho|_H$ into irreducible subrepresentations, to which the Stone-von Neumann theorem applies. Hence, the unique construction of V_1, given by the Stone-von Neumann theorem, allows us to compute the dimension of V_1. The lower bound is obtained by calculating the size of the orbit of the action of the normalizer of H on (σ_1, V_1).

In future, in order to understand the representations of the n-fold covering groups of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ completely, one needs to complete the classification of the decomposition of the reducible principal series representations, as well as construct the supercuspidal representations. There are also many questions to answer about their K-type decomposition, including investigating the behaviour of the representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ when the condition $n|q-1$, which implies the splitting of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ over $\mathrm{SL}_2(\mathcal{O})$, is relaxed.

Along the same vein, on the one hand, one can calculate the problem of finding reducibility points of $\pi_{\mathbf{s}}$ when n is even. Moreover, one can investigate this problem when χ is a ramified character. On the other hand, It is possible to extend my results on the distribution of K-types, explained above, to the entire K-type decomposition given in Theorem 1.

This thesis is organized as follows. In Chapter 1, we present the basic background of representation theory and the Stone-von Neumann theorem needed in both parts of the thesis. In Chapter 2, we cover the basics on central extensions, and the n-th Hilbert symbol, needed for Part I. Chapter 3 is devoted to a detailed construction of the n-fold covering groups of $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{GL}_2(\mathbb{F})$ by means of Kubota's cocycle, and the description of some of their subgroups. In Chapter 4, we study and solve the K-type problem for principal series representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$. In Chapter 5, we calculate the reducibility points for the unramified principal series representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$, and Chapter 6 is on the K-type distribution within the reducible unitary unramified principal series representation of the covering group. In Chapter 7, we present a verbatim copy of a joint paper on the minimal degree problem for the adjoint Chevalley groups G over $\mathcal{O}/\mathfrak{p}$, which has been submitted for publication.

Chapter 1

Background

We assume that the reader is familiar with basic theory of non-Archimedean local fields, and basic Lie algebra. For full exposition on these subjects see [Ser79a], [Cas86], and [Hum78]. In this section, we cover two main topics that are background to both parts of the thesis. In Section 1.1 we recall, without proof, some of the main concepts and results in harmonic analysis and representation theory of certain category of groups, which we need as tools to prove our results. Our main references for this part are [Tai75], [Sal98] and [BH06]. Section 1.2 is devoted to a detailed proof of one of the main ingredients of this thesis, a version of the Stone-von Neumann Theorem 1.2.2.

1.1 Representations of Locally Profinite Groups

1.1.1 Locally Profinite Groups

The class of groups we are interested in are *locally profinite groups.* Throughout this section we assume G is a locally profinite group.

Definition 1.1.1. *A* locally profinite group *is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup.*

For example, any finite group is locally profinite. We can generate a more interesting set of examples as follows. Let $\mathbb{F}$ be a non-Archimedean local field with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. Let $\kappa := \mathcal{O}/\mathfrak{p}$ be the residue field and $q = |\kappa|$ be its cardinality. Let $\mathcal{O}^\times$ denote the group of units in $\mathcal{O}$. Let ϖ be a uniformizing element of $\mathfrak{p}$. Every element in $\mathbb{F}^\times$ can be written uniquely as $x = a\varpi^m$ for a unit element $a \in \mathcal{O}^\times$ and an integer m, called the valuation of x and denoted by $\mathrm{val}(x)$. The field $\mathbb{F}$ carries an absolute value defined by $|x| = q^{-m} = q^{-\mathrm{val}(x)}$, for $x \neq 0$, and $|0| = 0$. Then $\mathcal{O} = \{a \in \mathbb{F} \mid \mathrm{val}(a) \geq 0\}$, $\mathcal{O}^\times = \{a \in \mathcal{O} \mid \mathrm{val}(a) = 0\}$ and $\mathfrak{p} = \{a \in \mathcal{O} \mid \mathrm{val}(a) > 0\}$.

The field of p-adic numbers $\mathbb{Q}_p$ for a prime p, and the field of formal Laurent series $\mathbb{R}((x))$, where $\mathbb{R}$ is a field, are non-Archimedean local fields.

Example 1.1.2. *The additive group of $\mathbb{F}$ is a locally profinite group. Set $\mathfrak{p}^m = \varpi^m\mathcal{O}$, for $m \in \mathbb{Z}$. Then the $\mathfrak{p}^m$ are open compact subgroups of $\mathbb{F}$ and give a fundamental system of open neighbourhoods of 0. The additive group $\mathbb{F}$ is the union of its compact open subgroups.*

Example 1.1.3. *The multiplicative group $\mathbb{F}^\times$ is a locally profinite group. Namely, the groups $1 + \mathfrak{p}^n$, $n \geq 1$, are compact open and give a fundamental system of open neighbourhoods of 1. Note that, unlike the additive group $\mathbb{F}$, the multiplicative group $\mathbb{F}^\times$ is not the union of its compact open subgroups. It is not difficult to see that $\mathbb{F}^\times \cong \mathbb{Z} \times \mathcal{O}^\times$.*

Example 1.1.4. *The group $\mathrm{SL}_2(\mathbb{F})$ of matrices with determinant one and matrix entries in $\mathbb{F}$ is a locally profinite group. The subgroups $K := \mathrm{SL}_2(\mathcal{O})$, matrices with determinant one and matrix entries in $\mathcal{O}$, and $K_j := \{g \in K \,|\, g \equiv I_2 \mod \mathfrak{p}^j\}$, $j \geq 1$, are compact open and give a fundamental system of open neighbourhoods of I_2. Note that K is a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{F})$.*

Remark 1.1.5. There exists a (unique up to a positive scalar) right-invariant Haar measure μ_G on G, which amounts to the property that for every locally constant compactly supported function f on G, and $g \in G$, $\int_G f(xg)d_{\mu_G}x = \int_G f(x)d_{\mu_G}x$. That is, $d_{\mu_G}(xg) = d_{\mu_G}(x)$ for all $x \in G$ and $g \in G$. Moreover, there exists a unique homomorphism $\delta_G : G \to \mathbb{R}^+$ such that $d_{\mu_G}(gx) = \delta_G(g)d_{\mu_G}(x)$. The function δ_G is called the modular character of G. We say G is unimodular if δ_G is trivial. Abelian groups and compact groups are unimodular. Throughout this thesis, we set $dx = d_{\mu_G}x$.

1.1.2 Some Harmonic Analysis on Locally Profinite Groups

Definition 1.1.6. *A* character *of the group G is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$. It is customary to write* 1 *for the trivial character of G. Moreover, χ is* unitary *if its image is contained in the unit circle.*

Note that characters of compact groups are unitary. If G is abelian, we denote the set of unitary characters of G by $\hat{G}$. It is well-known that $\hat{G}$ is itself an abelian group under point-wise multiplication and $\hat{\hat{G}} \cong G$ (Pontryagin duality).

Definition 1.1.7. *A character of the additive group $\mathbb{F}$ is called an* additive character *of $\mathbb{F}$. A character of the multiplicative group $\mathbb{F}^\times$ is called a* multiplicative character *of $\mathbb{F}^\times$.*

Because $\mathbb{F}$ is a union of compact groups, all its additive characters are unitary. However, a multiplicative character of $\mathbb{F}$ is not necessarily unitary. For example the character $x \mapsto |x|$ assumes arbitrarily large values and hence is not unitary. The proof of the following proposition can be found in [BH06, Proposition 1.7].

Proposition 1.1.8. *Let ψ be a non-trivial additive character of $\mathbb{F}$. The map*

$$\mathbb{F} \to \hat{\mathbb{F}} \quad a \mapsto \psi_a$$

where $\psi_a(x) = \psi(ax)$ is a group isomorphism.

Definition 1.1.9. *The* level *of an additive character ψ is the least integer d such that $\mathfrak{p}^d \subset \ker(\psi)$.*

Note that the continuity assumption on characters guaranties the existence of the level. For the rest of this document, we fix a choice λ of an additive character of level zero. That is, λ is trivial on $\mathcal{O}$ and not trivial on $\mathfrak{p}^{-1}$.

Remark 1.1.10. Since $\mathbb{F}^\times \cong \mathbb{Z} \times \mathcal{O}^\times$, for every $x \in \mathbb{F}^\times$, $x = x_0 \varpi^{\mathrm{val}(x)}$, $x_0 \in \mathcal{O}^\times$ and $\mathrm{val}(x) \in \mathbb{Z}$. Hence, any multiplicative character χ can be factored as $\chi(x) = \chi^*(x_0)|x|^s$ for all $x \in \mathbb{F}^\times$, where $s \in \mathbb{C}$ and χ^* is a character of $\mathcal{O}^\times$. In fact, $\chi^* = \chi|_{\mathcal{O}^\times}$. Moreover, we can assume that $\frac{-\pi}{\log q} < \mathrm{Im}(s) \leq \frac{\pi}{\log q}$.

Definition 1.1.11. *A multiplicative character χ is called* unramified *if $\chi|_{\mathcal{O}^\times} = 1$. Let m be the least positive integer such that $1 + \mathfrak{p}^m \subset \ker(\chi)$; then we say χ is* primitive *mod m, or* ramified *of degree m, or of* depth *$m-1$.*

We normalize the unimodular additive Haar measure $\mu_{\mathbb{F}}$ on $\mathbb{F}$ so that $\int_{\mathcal{O}} dx = 1$. The following lemma is useful later in this document.

Lemma 1.1.12. *Let $\psi : \mathbb{F} \to \mathbb{C}^\times$ be an additive character of level d and let $m < d$. Then*

$$\int_{\mathfrak{p}^m} \psi(x)dx = 0.$$

Proof: Choose $y \in \mathfrak{p}^m$ such that $\psi(y) \neq 1$. Since dx is an additive Haar measure, we may make the change of variable $x \to y + x$.

$$\int_{\mathfrak{p}^m} \psi(x)dx = \int_{\mathfrak{p}^m} \psi(y+x)dx = \psi(y) \int_{\mathfrak{p}^m} \psi(x)dx,$$

whence $\int_{\mathfrak{p}^m} \psi(x)dx = 0$. ∎

Definition 1.1.13. *Let $C_c^\infty(\mathbb{F})$ denote the space of complex-valued, compactly supported locally constant functions on $\mathbb{F}$.*

Definition 1.1.14. *Let λ be the (fixed) character of $\mathbb{F}$ of level zero. For $f \in L^1(\mathbb{F})$, define the Fourier transform of f to be the function $\hat{f} : \mathbb{F} \to \mathbb{C}$ given by*

$$\hat{f}(u) = \int_{\mathbb{F}} \lambda(-ux) f(x) dx \quad u \in \mathbb{F}.$$

The Fourier transform is bijective on $C_c^\infty(\mathbb{F})$ [Sal98] and its inverse is given by

$$f(u) = \int_{\mathbb{F}} \lambda(ux) \hat{f}(x) dx.$$

Note that for any $y \in \mathbb{F}^\times$, $d(yx) = |y|dx$. Hence, $d^\times x := dx|x|^{-1}$ is a multiplicative Haar measure on $\mathcal{O}^\times$. Our choice of normalization implies that $\int_{\mathcal{O}^\times} d^\times x = 1 - q^{-1}$. Let λ be the (fixed) additive character of $\mathbb{F}$ and let χ be a multiplicative character of $\mathbb{F}^\times$.

Definition 1.1.15. *A function $f : \mathbb{F} \to \mathbb{C}$ is* locally integrable *if it is integrable on each set $\{x \in \mathbb{F},\ q^{-m} \leq |x| \leq q^m\}$ for $m \geq 0$.*

Let f be a locally integrable function on $\mathbb{F}$. Define

$$f_m(x) = \begin{cases} f(x), & \text{if } q^{-m} \leq |x| \leq q^m \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.1.16. *The* principal value integral *of a locally integrable function f is defined by*

$$\mathrm{PV} \int_{\mathbb{F}} f(x) = \lim_{m \to \infty} \int f_m(x) dx. \tag{1.1.1}$$

A locally integrable function may not be integrable; however, if a locally integrable function f is indeed integrable, its integral is equal to the principal value integral of f [Tai75, Chapter 4]. Next, we will follow [Sal98] closely, to state the definition and values of Gamma and Bessel functions that are defined using the principal value integrals. These functions appear in the study of the intertwining operators of certain representations in Chapter 6. Recall that, for all $x = x_0 \varpi^{\mathrm{val}(x)}$ in $\mathbb{F}^\times$, $x_0 \in \mathcal{O}^\times$, $\chi(x) = \chi_*(x_0)|x|^s$, where χ_* is a character of $\mathcal{O}^\times$ and $s \in \mathbb{C}$.

Definition 1.1.17. *For a (non-trivial) character* $\chi = \chi_*|\cdot|^s$, $s \in \mathbb{C}$ *of* $\mathbb{F}^\times$, *the Gamma function* $\Gamma(\chi) = \Gamma_{\chi_*}(s)$ *is defined as follows:*

1. *If* χ *is ramified, let*

$$\Gamma(\chi) = \Gamma_{\chi_*}(s) = \mathrm{PV}\int_{\mathbb{F}} \lambda(x)\chi(x)\frac{dx}{|x|}.$$

2. *If* χ *is unramified and* $\mathrm{Re}(s) > 0$ *let*

$$\Gamma(\chi) = \Gamma_1(s) = \mathrm{PV}\int_{\mathbb{F}} \lambda(x)|x|^{s-1}dx.$$

3. *If* χ *is unramified and* $\mathrm{Re}(s) \leq 0$, $\frac{-\pi}{\log q} < \mathrm{Im}(s) \leq \frac{\pi}{\log q}$, *and* $s \neq 0$, *let* $\Gamma_1(s)$ *be the extension of* Γ_1 *by analytic continuation.*

Next, we state a lemma from [Sal98, Lemma 10.2], and a theorem [Sal98, Theorem 10.3] that gives the values for the Gamma function. In particular, it is shown in [Sal98] that the principal value integrals in Definition 1.1.1 converge.

Lemma 1.1.18. *Let* χ *be a ramified character with ramification degree* $h \geq 1$. *If* $|u| \neq q^h$ *then*

$$\int_{\mathcal{O}^\times} \lambda(ux)\chi(x)dx = 0.$$

Theorem 1.1.19. *Let* $\chi = \chi_*|\cdot|^s$ *be a character of* $\mathbb{F}^\times$.

1. *If* χ *is ramified of degree* $h \geq 1$, *then there exists* $C_{\chi_*} \in \mathbb{C}$ *such that* $\Gamma_{\chi_*}(s) = C_{\chi_*}q^{h(s-\frac{1}{2})}$, *where* $|C_{\chi_*}| = 1$ *and* $C_{\chi_*^{-1}}C_{\chi_*} = \chi_*(-1)$.

2. *If* χ *is unramified* ($\chi_* = 1$), *and* $s \neq 0$, *then*

$$\Gamma_1(s) = \frac{1-q^{s-1}}{1-q^{-s}}.$$

3. *For all nontrivial characters* χ *except for* $\chi(x) = |x|$, $\Gamma_{\chi_*}(s) = \chi_*(-1)\overline{\Gamma_{\chi_*^{-1}}(\bar{s})}$, *and* $\Gamma_{\chi_*}(s)\Gamma_{\chi_*^{-1}}(1-s) = \chi_*(-1)$, *and so* $\Gamma_{\chi_*}(s)\overline{\Gamma_{\chi_*}(1-\bar{s})} = 1$.

Another closely related function that will appear later in this thesis is the p-adic *Bessel function*. Next, we define this function, and state a theorem taken from [Sal98, Theorem 10.12 and 10.13], which gives us the values for the Bessel function.

Definition 1.1.20. *For a unitary character $\chi : \mathbb{F}^\times \to \mathbb{C}^\times$ and $u, v \in \mathbb{F}^\times$, the* Bessel function $J_\chi(u,v)$ *is defined as follows:*

$$J_\chi(u,v) = \mathrm{PV} \int_{\mathbb{F}} \lambda(ux + vx^{-1})\chi(x)\frac{dx}{|x|}.$$

Theorem 1.1.21. *Let χ be a non-trivial unitary character of $\mathbb{F}^\times$ and $u, v \in \mathbb{F}^\times$. If χ is unramified and $|uv| \leq q$, or if χ is ramified of degree h and $|uv| \leq q^h$, then*

$$J_\chi(u,v) = \chi(v)\Gamma(\chi^{-1}) + \chi^{-1}(u)\Gamma(\chi).$$

Note that we only include the cases that will be used in this thesis. The original theorem gives the values of the Bessel function completely.

1.1.3 Smooth Representations

A complex representation (π, V) of a locally profinite group G is a group homomorphism $\pi : G \to \mathrm{Aut}(V)$, where V is a complex vector space. Throughout this thesis, all representations of locally profinite groups are over complex vector spaces.

Definition 1.1.22. *A representation (π, V) of G is* smooth *if, for every $v \in V$, there exists a compact open subgroup K of G (depending on v) such that $\pi(g)v = v$, for all $g \in K$.*

Example 1.1.23. *A character of G is a smooth one-dimensional representation of G.*

For every subgroup H of G, let V^H denote the subspace of H-fixed vectors. That is, V^H is the space of all $v \in V$ such that $\pi(g)v = v$, for all $g \in H$. Therefore, (π, V) is smooth if and only if V is the union of its K-fixed spaces, where K ranges over the compact open subgroups of G.

Definition 1.1.24. *A smooth representation* (π, V) *is* admissible *if* V^K *is finite-dimensional, for each compact open subgroup* K *of* G*.*

Definition 1.1.25. *Given two smooth representations* (π_1, V_1) *and* (π_2, V_2) *of* G*, an* intertwining operator $f : V_1 \to V_2$ *is a linear map that commutes with the action of* G*, that is,*

$$f \circ \pi_1(g) = \pi_2(g) \circ f \quad \forall g \in G.$$

The space of all intertwining operators from π_1 *to* π_2 *is denoted by* $\mathrm{Hom}_G(\pi_1, \pi_2)$*. We say* π_1 *and* π_2 *are* equivalent *or* isomorphic *if there exists a bijective linear map in* $\mathrm{Hom}_G(\pi_1, \pi_2)$*.*

Definition 1.1.26. *A representation* (π, V) *of* G *is called* irreducible *if* V *admits no non-trivial* G*-stable subspaces.*

The next lemma, shows that all of the intertwining operators from an irreducible representation to itself are multiples of the identity operator.

Lemma 1.1.27 (Schur's Lemma)**.** *If* (π, V) *is an irreducible smooth representation of* G*, then* $\mathrm{End}_G(V) := \mathrm{Hom}_G(\pi, \pi) \cong \mathbb{C}$*.*

It follows directly from Schur's lemma that if (π, V) is an irreducible representation of G, then the center of G acts by a character. We call this character the *central character* of π.

A smooth representation of a locally profinite group G is not necessarily G-semisimple; that is, it might not be completely decomposable into a direct sum of its irreducible constituents. However, (π, V) is K-semisimple for any compact open subgroup K of G. More precisely, we have the following proposition, proven in [BH06].

Proposition 1.1.28. *Let* G *and* K *be a locally profinite group and an open compact locally profinite group respectively. Then the following statements hold.*

1. *Every irreducible representation of* K *is finite-dimensional.*

2. *If $K \subset G$, and (π, V) is a smooth representation of G, then V is a direct sum of its irreducible K-subspaces.*

3. *If G is abelian, any irreducible smooth representation of G is one-dimensional.*

Let (σ, W) be a smooth representation of a closed subgroup H of G. Define $\mathrm{Ind}_H^G\sigma$ to be the vector space of functions $f : G \to W$ satisfying

- $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and $g \in G$,
- there exists a compact open subgroup K of G, depending on f, such that $f(gx) = f(g)$ for all $x \in K$ and $g \in G$.

We construct a representation of G over the vector space $\mathrm{Ind}_H^G\sigma$ by letting G act by right translation. That is, $(g \cdot f)(x) = f(xg)$ for all $x, g \in G$, $f \in \mathrm{Ind}_H^G\sigma$. By definition, the G-representation $\mathrm{Ind}_H^G\sigma$ is a smooth representation of G.

Definition 1.1.29. *Define the* induced representation *of G by σ to be the space $\mathrm{Ind}_H^G\sigma$ together with the right translation action of G.*

Remark 1.1.30. When G is unimodular, sometimes it is beneficial to consider the *normalized* induction $\mathrm{Ind}_H^G\delta_H^{1/2}\sigma$ instead, where δ_H is the modular character of H. Set $\mathrm{ind}_H^G\sigma := \mathrm{Ind}_H^G\delta_H^{1/2}\sigma$.

Remark 1.1.31. It is customary to use $\mathrm{Ind}_H^G\sigma$ to indicate the induction space together with the right translation action. There are instances in this document where we consider the vector space $\mathrm{Ind}_H^G\sigma$ (or $\mathrm{ind}_H^G\sigma$) together with a different action. To avoid confusion, only in those instances, we will denote the vector space by $\mathcal{E}(\mathrm{Ind}_H^G\sigma)$ (or $\mathcal{E}(\mathrm{ind}_H^G\sigma)$).

Lemma 1.1.32. *Let G be a locally profinite group and let H be a closed subgroup of G such that $H\backslash G$ is compact. Let (σ, V) be an admissible representation of H. Then $\mathrm{Ind}_H^G\sigma$ is admissible.*

Proof: Let K_0 be an open compact subgroup of G. We want to show that $(\mathrm{Ind}_H^G\sigma)^{K_0}$ is finite-dimensional. Since $H\backslash G$ is compact, the double coset space $H\backslash G/K_0$ is finite. Let $\{g_1, \ldots, g_r\}$ be a set of double coset representatives of $H\backslash G/K_0$. Every $g \in G$ can be written as $g = hg_ik$ for some $1 \leq i \leq r$, $h \in H$ and $k \in K_0$. Hence, for all $f \in (\mathrm{Ind}_H^G\sigma)^{K_0}$,

$$f(g) = f(hg_ik) = f(hg_i) = \sigma(h)f(g_i),$$

so f is uniquely determined by its values on $\{g_1, \ldots, g_r\}$, which are vectors in V. Consider the compact open subgroup $H \cap g_iK_0g_i^{-1}$ of G. Observe that for $h \in H \cap g_iK_0g_i^{-1}$, we have $h = g_ikg_i^{-1}$ for some $k \in K_0$ and hence $hg_i = g_ik$. Therefore

$$\sigma(h)f(g_i) = f(hg_i) = f(g_ik) = f(g_i).$$

Therefore, $f(g_i) \in V^{H\cap g_iK_0g_i^{-1}}$. Note that since σ is admissible, for $1 \leq i \leq r$, $V^{H\cap g_iK_0g_i^{-1}}$ is finite-dimensional. Hence the image of g_i under f is in a finite-dimensional subspace of V. Therefore, $\left(\mathrm{Ind}_H^G\sigma\right)^{K_0}$ is spanned by finitely many functions. ∎

For a representation π of G and a subgroup H of G, we denote the restriction of π to H by $\mathrm{Res}_H\pi$. The next proposition [BH06, Proposition 2.4], which relates induction and restriction, gives a fundamental property of the induced representations.

Proposition 1.1.33 (Frobenius Reciprocity)**.** *Let H be a closed subgroup of G. For a smooth representation (σ, W) of H and a smooth representation (π, V) of G, we have the following isomorphism:*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\sigma) \cong \mathrm{Hom}_H(\mathrm{Res}_H\pi, \sigma).$$

For a subgroup H of a group G and $x \in G$, set $H^x = \{x^{-1}hx \mid h \in H\}$. If (σ, W) is a representation of H, the *conjugate representation* $\sigma^x : H^{x^{-1}} \to \mathrm{Aut}(W)$ defined by $\sigma^x(g) = \sigma(x^{-1}gx)$ is a representation of $H^{x^{-1}}$. The following proposition is a version of a well-known result originally due to Mackey [Mac52].

Theorem 1.1.34 (Mackey's Theorem)**.** *Let G be a locally profinite group and let H and L be closed subgroups of G such that $L\backslash G/H$ is finite. Let (σ, W) be an admissible representation of H and let S be a finite set of double coset representatives for $L\backslash G/H$. Then*

$$\mathrm{Res}_L \mathrm{Ind}_H^G \sigma = \bigoplus_{s\in S} \mathrm{Ind}_{H^{s^{-1}}\cap L}^L \sigma^s.$$

Next, we state a result that relates the intertwining maps between induced representations of finite groups, with functions in the corresponding *Hecke algebras.* Hecke algebras can be defined in the more general setting of locally profinite groups [BH06]. However, we only state the finite group version, which suffices for our purposes.

Definition 1.1.35. *Let G be a finite group. Let $\mathcal{H}(G)$ be the vector space of all functions $\phi : G \to \mathbb{C}$. For $\phi_1, \phi_2 \in \mathcal{H}(G)$, the convolution $\phi_1 * \phi_2$, given by*

$$(\phi_1 * \phi_2)(g) = \frac{1}{|G|}\sum_{x\in G} \phi_1(x)\phi_2(x^{-1}g), \quad \forall g \in G,$$

equips $\mathcal{H}(G)$ with the structure of an associative $\mathbb{C}$-algebra.

The algebra $\mathcal{H}(G)$ is isomorphic to the group algebra $\mathbb{C}[G]$.

Definition 1.1.36. *Let L and H be subgroups of a finite group G and let $\rho : L \to \mathbb{C}$ and $\chi : H \to \mathbb{C}$ be characters of L and H respectively. Define the vector space*

$$\mathcal{H}(L\backslash G/H, \rho, \chi) = \{f : G \to \mathbb{C} \mid f(lgh) = \rho(l)f(g)\chi(h),\ l \in L, h \in H\}.$$

Definition 1.1.37. *For a subgroup L of a finite group G the* Hecke algebra *$\mathcal{H}_L$ is the pair $(\mathcal{H}(L\backslash G/L, 1, 1), *)$.*

The proof of the next proposition can be found in [Nev05, Proposition 3.2]

Proposition 1.1.38. *Let L and H be subgroups of a finite group G and let $\rho : L \to \mathbb{C}$ and $\chi : H \to \mathbb{C}$ be characters of L and H respectively. The intertwining operators from $\mathrm{Ind}_H^G\chi$ to $\mathrm{Ind}_L^G\rho$ are given by convolution with functions in $\mathcal{H}(L\backslash G/H,\ \rho, \chi)$.*

Suppose the locally profinite group G is the group of $\mathbb{F}$-rational points of a reductive linear algebraic group defined over $\mathbb{F}$ (or more generally, a finite topological covering of such a group, as for example the group described in Chapter 3). Let P be a parabolic subgroup of G with unipotent radical N, Levi subgroup M and Levi decomposition $P = MN$.

Definition 1.1.39. *Let (π, V) be a smooth representation of G. Define $V(N)$ to be the subspace of V generated by $\{\pi(n)v - v \mid n \in N, v \in V\}$. The space $V_N := V/V(N)$ inherits a smooth representation π_N of $P/N = M$. We call (π_N, V_N) the* Jacquet module *of (π, V) at N.*

Later in this thesis, we generalize the notion of Jacquet module to non-linear covers of a locally profinite group. The following lemma in taken from [BH06, Chapter 3.8].

Lemma 1.1.40. *Let (π, V) be a representation of G. Then a vector $v \in V$ lies in $V(N)$ if and only if there exists a compact open subgroup N_0 of N such that*

$$\int_{N_0} \pi(n)v dn = 0.$$

The next lemma is a direct implication of Proposition 1.1.33.

Lemma 1.1.41. *Let (σ, W) be a smooth representation of M. Extend σ trivially on N to obtain a representation of P, and let (π, V) be a smooth representation of G. Then*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_M(\pi_N, \sigma). \tag{1.1.2}$$

Next we prove some general representation theory facts that we will use later in the thesis.

Lemma 1.1.42. *Let A and B be locally profinite groups such that $A \subset B$. Moreover, let χ_A and χ_B be characters of A and B respectively. The character χ_B occurs as a constituent of $\mathrm{Ind}_A^B \chi_A$ if and only if $\mathrm{Res}_A \chi_B = \chi_A$.*

Proof: The character χ_B occurs in $\mathrm{Ind}_A^B \chi_A$ if and only if there exists an intertwining operator from χ_B to $\mathrm{Ind}_A^B \chi_A$. By Frobenius reciprocity

$$\mathrm{Hom}_B(\chi_B, \mathrm{Ind}_A^B \chi_A) \cong \mathrm{Hom}_A(\mathrm{Res}_A \chi_B, \chi_A).$$

Since $\mathrm{Res}_A \chi_B$ and χ_A are irreducible, it follows from Schur's lemma that $\mathrm{Res}_A \chi_B = \chi_A$. ∎

Lemma 1.1.43. *Let A and B be abelian groups such that $A \subset B$ and $[B : A] = l < \infty$. Moreover, let χ be a character of A. Then $\mathrm{Ind}_A^B \chi$ decomposes as a direct sum of l distinct characters.*

Proof: Since B is abelian, the l-dimensional representation $\mathrm{Ind}_A^B \chi$ decomposes into a direct sum of l characters. To see that the characters are distinct, note that by Frobenius reciprocity,

$$\mathrm{Hom}_B(\mathrm{Ind}_A^B \chi, \mathrm{Ind}_A^B \chi) \cong \mathrm{Hom}_A(\mathrm{Res}_A \mathrm{Ind}_A^B \chi, \chi).$$

By Mackey's theorem

$$\mathrm{Res}_A \mathrm{Ind}_A^B \chi = \bigoplus_{s \in S} \mathrm{Ind}_{A \cap {}^s A}^A \chi^s, \tag{1.1.3}$$

where S is a set of coset representatives for B/A. Hence, $|S| = l$. Since B is abelian, ${}^s A = A$ and $\chi^s = \chi$ for all $s \in S$. Hence, (1.1.3) simplifies to $\chi^{\oplus l}$. Therefore,

$$\dim \mathrm{Hom}_B(\mathrm{Ind}_A^B \chi, \mathrm{Ind}_A^B \chi) \;=\; \dim \mathrm{Hom}_A(\chi^{\oplus l}, \chi) = l.$$

So, $\mathrm{Ind}_A^B \chi$ decomposes as a direct sum of l distinct characters. ∎

1.2 Stone-von Neumann Theorem

The Stone-von Neumann theorem characterizes the irreducible representations of *Heisenberg groups* in terms of their central characters. Here we state a version of the Stone-von Neumann theorem based on [McN12]. Then we elaborate on the proof given in the same paper. Note that one can see a variety of analogues of the Stone-von Neumann theorem in the literature. The origins and development of the theorem as well as its variants are discussed in [Ros04].

1.2.1 Definitions

Recall that for any group G, we define the lower central series of G inductively as follows. Set

$$Z_0(G) = \{1\}, \;\; Z_1(G) = Z(G),$$

and for each $i > 0$, set $Z_{i+1}(G)$ to be the preimage in G of the centre of $G/Z_i(G)$ under the natural projection. A group is called nilpotent if $Z_c(G) = G$ for some integer $c \geq 0$. The smallest such c is called the nilpotence class of G.

Definition 1.2.1. *A Heisenberg group is defined to be any group with nilpotence class two.*

Let H be a Heisenberg group and let A be a maximal abelian subgroup of H. Moreover, let χ' be a character of A and let $\mathrm{Ind}_A^H \chi' = \{f : H \to \mathbb{C} \mid f(ah) = \chi'(a)f(h) \text{ for all } a \in A, h \in H\}$ be the induced representation, where H acts by right translation.

Theorem 1.2.2 (Stone-von Neumann)**.** *Let H be a Heisenberg group with center $Z(H)$ such that $H/Z(H)$ is finite, and let χ be a character of $Z(H)$. Suppose that* $\ker(\chi) \cap [H, H] = \{1\}$*. Then there is a unique (up to isomorphism) irreducible representation π of H with central character χ. Let A be any maximal abelian subgroup of H and let χ' be any extension of χ to A. Then $\pi \cong \mathrm{Ind}_A^H \chi'$.*

1.2.2 Some Lemmas

Lemma 1.2.3. *A non-abelian group H is a Heisenberg group if and only if $Z(H)$ contains the commutator subgroup of H.*

Proof: Observe that

$$\begin{aligned} Z(H/Z(H)) &= \{\overline{a} \in H/Z(H) \mid [\overline{a}, \overline{g}] = 1 \;\forall \overline{g} \in H/Z(H)\} \\ &= \{\overline{a} \in H/Z(H) \mid [a, g]Z(H) = Z(H) \;\forall g \in H \text{ and } \forall a \in \overline{a}\}. \end{aligned}$$

So, $Z_2(H) = \{a \in H \mid [a, g] \in Z(H) \;\forall g \in H\}$. Suppose H is a Heisenberg group. By definition, $Z_2(H) = H$, which implies $[a, g] \in Z(H)$ for all $a, g \in H$. That is, $[H, H] \subseteq Z(H)$. Conversely, $[H, H] \subseteq Z(H)$ implies $Z_2(H) = H$, which means H is Heisenberg. ∎

Lemma 1.2.4. *Let A be a maximal abelian subgroup of a Heisenberg group H. Then A is normal in H.*

Proof: Note that $[H, H] \subseteq Z(H) \subseteq A$. Hence, for every $h \in H$ and $a \in A$, $[h, a^{-1}] = h^{-1}aha^{-1} \in [H, H] \subset A$, and therefore, $h^{-1}ah \in A$. ∎

Lemma 1.2.5. *Let H, A and χ' be as in the statement of Theorem 1.2.2. Suppose M is an invariant subspace of $\mathrm{Ind}_A^H \chi'$, and let f be in M such that $A \subsetneq \mathrm{Supp}(f)$. Then, there exists a function $f' \in M$ such that $A \subseteq \mathrm{Supp}(f') \subsetneq \mathrm{Supp}(f)$.*

Proof: Choose an element $h \in \mathrm{Supp}(f) \setminus A$. If h commutes with all the elements in A, then the subgroup generated by $\{h^{-1}\} \cup A$ is abelian, which contradicts the maximality of A. Therefore, there exists $a \in A$ such that $[a, h^{-1}] \neq 1$. By Lemma 1.2.3, $[a, h^{-1}] \in Z(H)$. Consider the function

$$f' = a \cdot f - \chi([a, h^{-1}])\chi'(a)f.$$

Since M is H-invariant, f' is in M. Note that since A is a normal subgroup in H, for all $x \in H$ we have

$$a \cdot f(x) = f(xa) = f(xax^{-1}x) = \chi'(xax^{-1})f(x).$$

Therefore, $\mathrm{Supp}(a \cdot f) \subseteq \mathrm{Supp}(f)$. Moreover, $\chi([a, h^{-1}])\chi'(a)f$ is a scalar multiple of f and hence $\mathrm{Supp}(f') \subseteq \mathrm{Supp}(f)$. Thus, proving that $f'(h) = 0$ implies that $\mathrm{Supp}(f')$ is strictly contained in $\mathrm{Supp}(f)$. Because $hah^{-1} \in A$, and χ' is an extension of χ to A, we have

$$\chi([a, h^{-1}])\chi'(a) = \chi'(a^{-1})\chi'(hah^{-1})\chi'(a) = \chi'(hah^{-1}).$$

and hence,

$$\left(a \cdot f - \chi([a, h^{-1}])\chi'(a)f\right)(h) = \chi'(hah^{-1})f(h) - \chi'(hah^{-1})f(h) = 0.$$

So, f' vanishes on h which shows its support is strictly smaller than that of f. Moreover, for any $a' \in A$, we have

$$f'(a') = f(a'a) - \chi([a, h^{-1}])f(aa') = \left(1 - \chi([a, h^{-1}])\right) f(aa')$$

is nonzero, because $A \subseteq \mathrm{Supp}(f)$ and $[a, h^{-1}] \notin \ker(\chi)$ as $\ker(\chi) \cap [H, H] = \{1\}$. Therefore, $A \subseteq \mathrm{Supp}(f')$. ∎

Lemma 1.2.6. *Let H be as in the statement of Theorem 1.2.2, and let (π, V) be an irreducible representation of H. Then π is finite-dimensional.*

Proof: Let $\mathcal{B} = \{h_1, \ldots, h_m\}$ be a complete set of representatives of H/Z and let $v \in V$ be a non-zero vector. Set $B = \{\pi(h_i)v \mid h_i \in \mathcal{B}\}$, and let W be the subspace of V spanned by B. Let χ be the central character of π. Every element v_0 of W can be expressed as $\sum_{i=1}^{m} \lambda_i \pi(h_i)v$ where $\lambda_i \in \mathbb{C}$. For all $h \in H$ and $v_0 \in W$ we have

$$\pi(h)v_0 = \pi(h)\sum_{i=1}^{m} \lambda_i \pi(h_i)v = \sum_{i=1}^{m} \lambda_i \pi(hh_i)v = \sum_{j=1}^{m} \lambda_i \pi(h_{j_i} z_j)v,$$

for $h_{ji} \in \mathcal{B}$ and some $z_j \in Z(H)$. So,

$$\pi(h)v_0 = \sum_{j=1}^{m} \lambda'_j \pi(h_j)v,$$

where $\lambda'_j = \lambda_j \chi(z_j)$. So, W is a non-trivial H-invariant subspace of V. Since π is irreducible, it implies that $W = V$. Hence, V is at most m-dimensional. ∎

1.2.3 Proof of the Stone-von Neumann Theorem

Proof: Let π be an irreducible representation of H with central character χ. By Lemma 1.2.6, π is finite-dimensional. The restriction $\mathrm{Res}_A^H \pi$ has an irreducible constituent χ', that is one-dimensional, because A is abelian, and $\chi'|_{Z(H)} = \chi$. Let ϕ be the projection homomorphism from $\mathrm{Res}_A^H \pi$ to χ'. Then ϕ is a non-trivial element in $\mathrm{Hom}_A(\mathrm{Res}_A^H \pi, \chi')$. By Frobenius reciprocity,

$$\mathrm{Hom}_A(\mathrm{Res}_A^H \pi, \chi') \cong \mathrm{Hom}_H(\pi, \mathrm{Ind}_A^H \chi').$$

Therefore, there exists a non-trivial $\phi' \in \mathrm{Hom}_H(\pi, \mathrm{Ind}_A^H \chi')$. Note that $\ker(\phi')$ is a proper invariant subspace of the irreducible π and hence trivial. Therefore, $\mathrm{Im}(\phi')$ is a non-trivial H-invariant subspace of $\mathrm{Ind}_A^H \chi'$. We want to show that ϕ' is an isomorphism, and all we need to show is that $\mathrm{Ind}_A^H \chi'$ is irreducible. Define a function $f_A : H \to \mathbb{C}$ by

$$f_A(x) = \begin{cases} \chi'(x) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, $f_A \in \mathrm{Ind}_A^H \chi'$ and the one-dimensional vector space $F = \mathbb{C} f_A$ is a subspace of $\mathrm{Ind}_A^H \chi'$. Moreover, it is not difficult to see that $H \cdot f_A$ generates $\mathrm{Ind}_A^H \chi'$. Let M be a non-trivial H-invariant subspace of $\mathrm{Ind}_A^H \chi'$. We show that M intersects F non-trivially and thus contains it. Choose a non-zero $f \in M$. After replacing f by

$h \cdot f$ for some $h \in H$, we can assume that $f(1) \neq 0$, so $A \subseteq \text{Supp}(f)$. If $A \neq \text{Supp}(f)$, by Lemma 1.2.5, there exists a $f' \in M$ such that $A \subseteq \text{Supp}(f') \subsetneq \text{Supp}(f)$. By continuing this process, in each step one coset of A in H is eliminated from the support of the resulting function. Since $|H/A|$ is finite, we can find a function in M whose support is equal to A and hence is in F. Because F is one-dimensional, this implies $F \subseteq M$. Hence, since M is H-invariant, $H \cdot f_A \subseteq M$. Therefore, $M = \text{Ind}_A^H \chi'$. Hence, $\text{Ind}_A^H \chi' \cong \pi$.

Next, we show that π is the unique representation of H, up to isomorphism, with central character χ. Precisely, for any extension χ'' of χ to A, we have $\text{Ind}_A^H \chi'' \cong \pi$. Let χ_1 and χ_2 be two distinct extensions of χ to A. Consider the function

$$\chi_1 \chi_2^{-1}(a) = \frac{\chi_1(a)}{\chi_2(a)}.$$

This is a character of A. Note that $\text{Res}_{Z(H)} \chi_1 = \text{Res}_{Z(H)} \chi_2 = \chi$. So $\chi_1 \chi_2^{-1}|_{Z(H)} = 1$, and hence, we can consider $\chi_1 \chi_2^{-1}$ as a character of $A/Z(H)$. Because H is a Heisenberg group, $H/Z(H)$ is abelian and hence, its irreducible representations are given by characters. So, $\chi_1 \chi_2^{-1}$ can be extended to a character of $H/Z(H)$. By Lemma 1.2.3, $[H, H] \subset Z(H)$ so that the map

$$\begin{aligned} H/Z(H) \times H/Z(H) &\rightarrow \mathbb{C} \\ (h_1 Z(H), h_2 Z(H)) &= \chi([h_1, h_2]) \end{aligned}$$

is well-defined. Note that $\chi([h_1, h_2]) = 1$ implies that $[h_1, h_2] \in \ker(\chi) \cap [H, H] = \{1\}$. For each $x \in H/Z(H)$, define

$$\begin{aligned} \chi_x : H/Z(H) &\rightarrow \mathbb{C} \\ hZ(H) &\rightarrow \chi([h, x]). \end{aligned}$$

This gives all the characters of $H/Z(H)$. Therefore, the extension of $\chi_1 \chi_2^{-1}$ to $H/Z(H)$ should be equal to χ_x for some x. So $\chi_1 \chi_2^{-1}(a) = \chi_x(a) = \chi([a, x])$, which implies that

$$\begin{aligned}\chi_1(a) &= \chi_2(a)\chi(a^{-1}x^{-1}ax) \\ &= \chi_2(a)\chi_2(a^{-1})\chi_2(x^{-1}ax) = \chi_2(x^{-1}ax).\end{aligned}$$

Hence, χ_1 and χ_2 are conjugate. Thus, the map $T : \mathrm{Ind}_A^H \chi_1 \to \mathrm{Ind}_A^H \chi_2$ defined by $T(f)(h) = f(x^{-1}h)$ is an H-intertwining operator, which implies $\mathrm{Ind}_A^H \chi_1 \cong \mathrm{Ind}_A^H \chi_2$. ∎

Part I

Representations of Metaplectic Covers of SL_2 over a p-adic Field

Chapter 2

Preliminaries

2.1 Central Extensions

Throughout this chapter, G and A denote groups such that A is abelian. Later on, in the thesis, we will specialize to the case $G = \mathrm{SL}_2(\mathbb{F})$, where $\mathbb{F}$ is a p-adic filed, and $A = \mu_n$, $n \geq 2$ is the group of n-th roots of unity.

Definition 2.1.1. *A central extension of G by A is a group $\widetilde{G}$ that fits in the short exact sequence*

$$1 \to A \to \widetilde{G} \to G \to 1, \tag{2.1.1}$$

where A, seen as a subgroup of $\widetilde{G}$, maps into the centre of $\widetilde{G}$.

Such a central extension need not be unique. In order to describe the structure of the group $\widetilde{G}$, we need to specify more data. To do so, there are two approaches. One approach is to specify a "section" of G in $\widetilde{G}$, and the other is to give a "2-cocycle" in the second cohomology group of G with coefficients in A. We will describe the above approaches and the link between them. First, let us give a few definitions.

Definition 2.1.2. *A* 2-cocycle*, or a* factor set*, is a function $\beta : G \times G \to A$ satisfying*

$$\beta(g_2, g_3)\beta(g_1, g_2g_3) = \beta(g_1, g_2)\beta(g_1g_2, g_3),$$

for all g_1, g_2 *and* g_3 *in* G*. If a* 2*-cocycle* $\beta : G \times G \to A$ *is of the form*

$$\beta(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}, \quad \textit{for all } g_1, g_2 \in G,$$

for a function $s : G \to A$*, we say* β *is a* 2-coboundary*.*

The set of 2-cocycles form a group under pointwise addition. The second cohomology group of G with coefficients in A, denoted by $H^2(G, A)$, is the quotient group of 2-cocycles modulo the subgroup of 2-coboundaries.

Given the central extension

$$1 \to A \xrightarrow{i} \widetilde{G} \xrightarrow{\mathsf{p}} G \to 1, \tag{2.1.2}$$

a *section* is a map $\mathfrak{s} : G \to \widetilde{G}$, such that $\mathsf{p} \circ \mathfrak{s}(g) = g$ for all $g \in G$. In fact, for any $g \in G$, $\mathfrak{s}$ chooses a class representative for $A\mathfrak{s}(g)$. Note that the section map $\mathfrak{s}$ is not required to send the identity element of G to the identity element of $\widetilde{G}$. However, if it does, it is called a *normalized section.* One can see that $\mathfrak{s}(g_1)\mathfrak{s}(g_2) \in A\mathfrak{s}(g_1g_2)$ for all $g_1, g_2 \in G$ and hence, there exists an element $\beta_{\mathfrak{s}}(g_1, g_2)$ in A such that $\mathfrak{s}(g_1)\mathfrak{s}(g_2) = \beta_{\mathfrak{s}}(g_1, g_2)\mathfrak{s}(g_1g_2)$. It is straightforward to show that $\beta_{\mathfrak{s}}$ is, in fact, a 2-cocycle. If $\mathfrak{s}$ is normalized, its corresponding 2-cocycle satisfies $\beta_{\mathfrak{s}}(1, g) = \beta_{\mathfrak{s}}(g, 1) = 1$ for all $g \in G$, and is called a *normalized 2-cocycle.* Moreover, β is independent from the choice of section up to a 2-coboundary. That is, if $\mathfrak{s}'$ is another choice of section, $\beta_{\mathfrak{s}'}(g_1, g_2) = \beta_{\mathfrak{s}}(g_1, g_2)\alpha(g_1, g_2)$, for all $g_1, g_2 \in G$, where $\alpha : G \to A$ is a 2-coboundary. Therefore, to any central extension $\widetilde{G}$ of G by A, we can associate a class in $H^2(G, A)$.

Remark 2.1.3. If the short exact sequence (2.1.2) is split, i.e., $\widetilde{G} = G \times A$, then there exists a section $\mathfrak{s} : G \to \widetilde{G}$, which is a group homomorphism. Moreover, the corresponding cocycle $\beta_{\mathfrak{s}}$ is trivial.

Definition 2.1.4. *We say two central extensions* $1 \to A \xrightarrow{i_1} \widetilde{G} \xrightarrow{p_1} G \to 1$ *and* $1 \to A \xrightarrow{i_2} \widetilde{G}' \xrightarrow{p_2} G \to 1$ *are* equivalent *if there exists a group isomorphism* $\phi : \widetilde{G} \to \widetilde{G}'$

such that the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i_1} & \widetilde{G} & \xrightarrow{p_1} & G & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle id} & & \Big|{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & A & \xrightarrow{i_1} & \widetilde{G}' & \xrightarrow{p_1} & G & \longrightarrow & 1 \end{array}$$

As is expected, equivalent central extensions correspond to the same class in $H^2(G, A)$.

Conversely, given a class in $H^2(G, A)$, we can construct the corresponding central extension. It is not difficult to see that every 2-cocycle is equivalent to a normalized 2-cocycle. We assume that the given class is represented by a normalized 2-cocycle β.

Definition 2.1.5. *Let β be a normalized 2-cocycle, and define $\widetilde{G}$ to be $G \times A = \{(g, \zeta) |\ g \in G,\ \zeta \in A\}$ as a set, and define the multiplication on $\widetilde{G}$ by*

$$(g_1, \zeta_1)(g_2, \zeta_2) \quad = \quad (g_1 g_2, \beta(g_1, g_2)\zeta_1\zeta_2). \tag{2.1.3}$$

Lemma 2.1.6. *The set $\widetilde{G}$ together with the multiplication given in* (2.1.3) *is a central extension of G by A.*

Proof: First, let us show that $\widetilde{G}$ is a group. Since the 2-cocycle β is normalized, the element $(1, 1) \in G \times A$ is the identity element of $\widetilde{G}$. Also, it is easy to check that for all $(g, a) \in \widetilde{G}$, $(g, a)^{-1} = (g^{-1}, \beta(g, g^{-1})^{-1}a^{-1})$. Let (g_i, a_i), $i = 1, 2, 3$ be three elements of $\widetilde{G}$. We compute

$$\begin{aligned} ((g_1, a_1)(g_2, a_2))\,(g_3, a_3) &= (g_1 g_2, \beta(g_1, g_2)a_1 a_2)(g_3, a_3) \\ &= (g_1 g_2 g_3, \beta(g_1, g_2)\beta(g_1 g_2, g_3)a_1 a_2 a_3) \\ &= (g_1 g_2 g_3, \beta(g_2, g_3)\beta(g_1, g_2 g_3)a_1 a_2 a_3) \quad \text{by 2-cocycle property} \\ &= (g_1, a_1)(g_2 g_3, \beta(g_2, g_3)a_2 a_3) \\ &= (g_1, a_1)\,((g_2, a_2)(g_3, a_3))\,. \end{aligned}$$

Thus, $\widetilde{G}$ is a group. The maps

$$i : A \to \widetilde{G}, \quad a \mapsto (1, a), \tag{2.1.4}$$

and

$$\mathsf{p} : \widetilde{G} \to G, \quad (g, a) \mapsto g. \tag{2.1.5}$$

are group homomorphisms, and so

$$1 \to A \xrightarrow{i} \widetilde{G} \xrightarrow{\mathsf{p}} G \to 1 \tag{2.1.6}$$

is a short exact sequence. To see $i(A)$ is central in $\widetilde{G}$, let $\widetilde{g} = (g, a')$ be an arbitrary element of $\widetilde{G}$. Then by (2.1.3), since β is normalized and A is abelian, we have

$$(1, a)(g, a') = (g, \beta(1, g)aa') = (g, \beta(g, 1)a'a) = (g, a')(1, a).$$

Hence, $i(A)$ is central in $\widetilde{G}$. By Definition 2.1.1, $\widetilde{G}$ is a central extension of G by A. ∎

Remark 2.1.7. Suppose G and A are topological groups. In order to obtain a topological covering, it suffices for the 2-cocycle β to be continuous in a neighbourhood of the identity.

Note that if $\beta : G \times G \to A$ is the trivial map, the construction in Definition 2.1.5 gives us the group $G \times A$. In fact, every function in the identity class of $H^2(G, A)$ results in a central extension equivalent to $G \times A$, which we call a *trivial extension.*

For any subgroup H of G, $\widetilde{H} := \mathsf{p}^{-1}(H)$ is a subgroup of $\widetilde{G}$, and in fact a central extension of H by A:

$$1 \to A \xrightarrow{i} \widetilde{H} \xrightarrow{\mathsf{p}} H \to 1. \tag{2.1.7}$$

Definition 2.1.8. *The central extension splits over H if $\widetilde{H} \cong A \times H$; that is $\beta|_{H \times H}$ is a 2-coboundary. The map $s : H \to A$ satisfying $\beta(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$ is called a splitting map.*

If $\beta|_{H\times H} = 1$, we say the central extension splits trivially over H.

Lemma 2.1.9. *Let H be a subgroup of G such that the central extension splits over H via the splitting map s. Then $\widetilde{H} \cong H \times A$ via the isomorphism given by*

$$\begin{aligned} \phi : \widetilde{H} &\to H \times A \\ (h,a) &\mapsto (h, s(h)a). \end{aligned}$$

Proof: First we show that ϕ is a group homomorphism. Let $(h_1,a_1),(h_2,a_2) \in \widetilde{H}$; then

$$\begin{aligned} \phi\left((h_1,a_1),(h_2,a_2)\right) &= \phi\left((h_1h_2, \beta(h_1,h_2)a_1a_2)\right) \\ &= \phi\left((h_1h_2, s(h_1)s(h_2)s(h_1h_2)^{-1}a_1a_2)\right) \\ &= (h_1h_2, s(h_1)s(h_2)a_1a_2) \\ &= (h_1, s(h_1)a_1)(h_2, s(h_2)a_2) \\ &= \phi\left((h_1,a_1)\right)\phi\left((h_2,a_2)\right). \end{aligned}$$

It is easy to see that ϕ is surjective. If $(h,a) \in \ker(\phi)$ then $h = 1$ and $s(1)a = 1$. The identity $1 = \beta(1,1) = s(1)s(1)s(1)^{-1}$ implies that $s(1) = 1$, and hence $a = 1$. Therefore, ϕ is an isomorphism. ∎

2.2 The n-th Hilbert Symbol

Let $\mathbb{F}$ be a p-adic field. Let $n \geq 2$ be an integer. Recall that $q = |\mathcal{O}/\mathfrak{p}|$. Throughout the rest of the document, we assume that $n|q-1$.

The n-th Hilbert symbol is a map $(\ ,\ )_n : \mathbb{F}^\times \times \mathbb{F}^\times \to \mu_n$. The following propositions list the fundamental properties of the n-th Hilbert symbol as well as a formula to calculate the symbol. For more detail on the n-Hilbert symbol, including the definition and the proof of the following propositions, see [Ser79b, Ch XIV].

Proposition 2.2.1. *For every α, β, γ in $\mathbb{F}^\times$, the following properties hold:*

(i) $(\alpha, \beta\gamma)_n = (\alpha, \beta)_n(\alpha, \gamma)_n$.

(ii) $(\alpha\beta, \gamma)_n = (\alpha, \gamma)_n(\beta, \gamma)_n$.

(iii) $(\alpha, -\alpha)_n = 1$.

(iv) $(\alpha, 1-\alpha)_n = 1$ *whenever* $\alpha \neq 1$.

(v) $(\alpha, \beta)_n = 1 \Leftrightarrow \beta$ *is the norm of an element in the extension* $\mathbb{F}(\alpha^{1/n})/\mathbb{F}$.

(vi) $(\alpha, \beta)_n(\beta, \alpha)_n = 1$.

(vii) $(\alpha, \beta)_n = 1$ *for every* $\alpha \in \mathbb{F}^\times$ *if and only if* $\beta \in \mathbb{F}^{\times n}$.

Since $n|q-1$, $\mu_n \subseteq \kappa^\times$ and we have the following easy formula for the n-th Hilbert symbol, proven in [Ser79b].

Proposition 2.2.2. *Let a and b be in $\mathbb{F}^\times$. Then,*

$$(a, b)_n = (\bar{c})^{\frac{q-1}{n}},$$

where

$$c = (-1)^{\text{val}(a)\text{val}(b)} \frac{a^{\text{val}(b)}}{b^{\text{val}(a)}}$$

and $\bar{c}$ denotes the image of c in $\kappa^\times$.

Remark 2.2.3. Note that $\text{val}(c) = 0$, and therefore $(\, , \,)_n$ is well-defined.

Remark 2.2.4. Note that $n|q-1$ if and only if $\mu_n \subset \mathbb{F}$ and $(p, n) = 1$. The n-th Hilbert symbol is defined, and the properties in Proposition 2.2.1 hold, if $\mathbb{F}$ contains μ_n. However, the formula given in Proposition 2.2.2 holds only if $n|q-1$.

Next, we calculate some useful identities, which are easy to see via Propositions 2.2.1 and 2.2.2, and that we use frequently in the computations involving the n-th Hilbert symbol.

Corollary 2.2.5. *Let α and β be in $\mathbb{F}^\times$. Then the following identities hold:*

(i) $(\alpha, 1)_n = (1, \alpha)_n = 1$.

(ii) $(\alpha, \beta)_n^{-1} = (\alpha, \beta^{-1})_n = (\alpha^{-1}, \beta)_n$.

(iii) $(\beta, \alpha^{-1})_n = (\alpha, \beta)_n$.

(iv) $(\alpha^{-1}, \beta^{-1})_n = (\alpha, \beta)_n$.

(v) $(\alpha, \alpha)^2 = 1$.

(vi) $(\frac{\alpha+\beta}{\alpha}, \frac{\alpha+\beta}{\beta})_n = 1$.

Proof: Part (i) follows from parts (i) and (ii) in Proposition 2.2.1. Part (ii) follows from applying part (i) of Proposition 2.2.1: $(\alpha, \beta)_n(\alpha, \beta^{-1})_n = (\alpha, 1)_n$ which is equal to one by (i). To see (iii), note that $(\beta, \alpha^{-1})_n = (\beta, \alpha)_n^{-1}$ by (ii), and that $(\beta, \alpha)_n^{-1} = (\alpha, \beta)_n$ by (vi) of Proposition 2.2.1. Part (iv) is a direct implication of (iii), and (v) is a consequence of (vi) in Proposition 2.2.1. Finally, to see part (vi), set $\gamma = 1 + \frac{\beta}{\alpha}$. Then, $\left(\frac{\alpha+\beta}{\alpha}, \frac{\alpha+\beta}{\beta}\right)_n = \left(\gamma, \frac{\gamma}{\gamma-1}\right)_n = (\gamma, -\gamma)_n(\gamma, 1-\gamma)_n$, which equals one by Proposition 2.2.1.

∎

Lemma 2.2.6. *Let $\alpha, \beta \in \mathbb{F}^\times$ be such that $\alpha + \beta \in \mathbb{F}^\times$. Then*

$$(\alpha, \beta)(-\beta\alpha^{-1}, \alpha + \beta) = 1.$$

Proof: By (iv) and (i) in Proposition 2.2.1 we have

$$1 = \left(-\frac{\beta}{\alpha}, 1 + \frac{\beta}{\alpha}\right)_n = \left(-\frac{\beta}{\alpha}, \frac{\alpha+\beta}{\alpha}\right)_n = \left(-\frac{\beta}{\alpha}, \alpha + \beta\right)_n \left(-\frac{\beta}{\alpha}, \frac{1}{\alpha}\right)_n.$$

Note that

$$\left(-\frac{\beta}{\alpha}, \frac{1}{\alpha}\right)_n = \left(\beta, \frac{1}{\alpha}\right)_n \left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right)_n = (\beta, \alpha)_n{}^{-1} = (\alpha, \beta)_n.$$

And hence the result. ∎

Lemma 2.2.7. *The following equalities hold.*

(i) $(\alpha,\beta)_n = 1$ *whenever* $\alpha,\beta \in \mathcal{O}^\times$.

(ii) $(\alpha,\beta)_n = 1$ *whenever* $\alpha \equiv 1 \bmod \mathfrak{p}$, *and* $\beta \in \mathbb{F}^\times$.

Proof: If $\alpha,\beta \in \mathcal{O}^\times$, then $\mathrm{val}(\alpha) = \mathrm{val}(\beta) = 0$, so the value of c in Proposition 2.2.2 is 1 and hence $(\alpha,\beta)_n = 1$. Note that for $\alpha \in 1+\mathfrak{p}$ and $\beta \in \mathbb{F}^\times$, $c = (-1)^0(1+\varpi x)^{\mathrm{val}(\beta)}$ for some $x \in \mathcal{O}$, and hence $\bar{c} = (1+\varpi x)^{\mathrm{val}(\beta)} = 1 \mod \mathfrak{p}$. ∎

When n is odd, the Hilbert symbol has additional nice properties as follows.

Lemma 2.2.8. *Let $n > 2$ be an odd integer. For each $\alpha,\beta \in \mathbb{F}^\times$, write $\alpha = \varpi^{\mathrm{val}(\alpha)}\alpha_0$ and $\beta = \varpi^{\mathrm{val}(\beta)}\beta_0$, where $\alpha_0,\beta_0 \in \mathcal{O}^\times$ and ϖ is the uniformizing element. Then the following identities hold:*

(i) $(\varpi,\varpi)_n = 1$,

(ii) $(\alpha,\varpi)_n = (\alpha_0,\varpi)_n = \overline{\alpha_0}^{\frac{q-1}{n}}$,

(iii) $(\alpha,\beta)_n = (\varpi,\beta_0)_n^{\mathrm{val}(\alpha)}(\varpi,\alpha_0)_n^{-\mathrm{val}(\beta),}$

(iv) $(\alpha,\alpha)_n = 1$.

Proof: Since n is odd, it follows that $\frac{q-1}{n}$ is even. This implies $(\varpi,\varpi)_n = 1$. To see part (ii), note that by Proposition 2.2.1, $(\alpha,\varpi)_n = (\varpi,\varpi)_n^{\mathrm{val}(\alpha)}(\alpha_0,\varpi)_n$, which is $(\alpha_0,\varpi)_n$ by part (i), and applying the formula in Proposition 2.2.2 gives $(\alpha_0,\varpi)_n = \overline{\alpha_0}^{\frac{q-1}{n}}$. Similarly, $(\alpha,\beta)_n$ decomposes as

$$(\alpha_0,\beta_0)_n(\varpi^{\mathrm{val}(\alpha)},\beta_0)_n(\alpha_0,\varpi^{\mathrm{val}(\beta)})_n(\varpi^{\mathrm{val}(\alpha)},\varpi^{\mathrm{val}(\beta)})_n. \tag{2.2.1}$$

Observe that, because $\alpha_0, \beta_0 \in \mathcal{O}^\times$, $(\alpha_0, \beta_0)_n = 1$, and by part (i) $(\varpi^{\mathrm{val}(\alpha)}, \varpi^{\mathrm{val}(\beta)})_n = 1$. So, by Proposition 2.2.1, (2.2.1) simplified to $(\varpi, \beta_0)_n^{\mathrm{val}(\alpha)}(\varpi, \alpha_0)_n^{-\mathrm{val}(\beta)}$, and hence part (iii). Finally, observe that by part (iii), $(\alpha, \alpha)_n = (\varpi, \alpha_0)_n^{\mathrm{val}(\alpha)}(\varpi, \alpha_0)_n^{-\mathrm{val}(\alpha)} = 1$. ∎

Remark 2.2.9. If n is even, then $(\varpi, \varpi)_n = \pm 1$ depending on the parity of $\frac{q-1}{n}$, and hence the analogous identities given in Lemma 2.2.8 are

(i) $(\varpi, \varpi)_n = (-1)^{\frac{q-1}{n}}$

(ii) $(\alpha, \varpi)_n = (-1)^{\frac{q-1}{n}\mathrm{val}(\alpha)}(\alpha_0, \varpi)_n$

(iii) $(\alpha, \beta)_n = (-1)^{\frac{q-1}{n}\mathrm{val}(\alpha)\mathrm{val}(\beta)}(\varpi, \beta_0)_n^{\mathrm{val}(\alpha)}(\varpi, \alpha_0)_n^{-\mathrm{val}(\beta)}$

(iv) $(\alpha, \alpha)_n = (-1)^{\frac{q-1}{n}\mathrm{val}(\alpha)}$.

Finally, let us state the Hensel's lemma [Kna07, Corollary 6.29], and demonstrate some applications of it that will be useful later in this thesis. Recall that a is a simple root of a polynomial $f(X)$ if $f(a) = 0$ and $f'(a) \neq 0$.

Lemma 2.2.10. *Let $f(X)$ be a polynomial in $\mathcal{O}[X]$. If $\bar{f}(X)$ is the reduced polynomial with coefficients in $\mathcal{O}/\mathfrak{p}$ and $\bar{a}$ is a simple root of $\bar{f}(X)$, then $f(X)$ has a simple root a in $\mathcal{O}$ whose image in $\mathcal{O}/\mathfrak{p}$ is $\bar{a}$.*

Lemma 2.2.11. *The group $1 + \mathfrak{p}$ is a subgroup of ${\mathbb{F}^\times}^n$.*

Proof: Let $\alpha \in 1 + \mathfrak{p}$. Consider the polynomial $f(X) = X^n - \alpha \in \mathcal{O}[X]$. Note that $f(1) = 0 \mod \mathfrak{p}$ and, since $(n, p) = 1$, $f'(1) \neq 0 \mod p$. Hence, by Hensel's lemma, $f(X)$ has a root in $\mathcal{O}$ and hence α is an n-th power. ∎

Lemma 2.2.12. *The index $[\mathcal{O}^\times : {\mathcal{O}^\times}^n]$ is n.*

Proof: Consider the homomorphism $\phi : \mathcal{O}^\times \to \mathcal{O}^{\times n}$. Note that $\ker(\phi) = \{x \in \mathcal{O}^\times |\, x^n = 1\}$. The equation $f(X) := X^n - 1 = 0$ has $(n, q-1)$ solutions in the cyclic group $\kappa^\times$, which is equal to n under our assumption of $n|q-1$. Let $\bar{x} \in \kappa^\times$ be any root of $f(X)$. Note that $f'(\bar{x}) = n\bar{x}^{n-1} \neq 0$, since $(n,p) = 1$. Hence, by Hensel's lemma, there exists a unique $x \in \mathcal{O}^\times$, such that $f(x) = 0$. Hence, $|\ker(\phi)| = n$. Therefore, $[\mathcal{O}^\times : \mathcal{O}^{\times n}] = |\ker(\phi)| = n$. ∎

Remark 2.2.13. If n is even, then one can similarly see that $[\mathcal{O}^\times : \mathcal{O}^{\times \frac{n}{2}}] = \frac{n}{2}$.

Chapter 3

Construction of Covering Groups of $\mathrm{SL}_2(\mathbb{F})$ and $\mathrm{GL}_2(\mathbb{F})$

From now until the end of part I of this thesis, we assume that $\mathbb{F}$ is a p-adic field, and $n > 2$ is an integer such that $n|q-1$. Set $\underline{n} = n$ if n is odd and $\underline{n} = \frac{n}{2}$ if n is even. Set $G = \mathrm{SL}_2(\mathbb{F})$ and $G' = \mathrm{GL}_2(\mathbb{F})$. Let I_2 denote the 2×2 identity matrix, set $\mathrm{dg}(t) := \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, $\mathrm{dg}(s,t) := \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right)$, $\mathrm{ut}(m) := \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$, and $\mathrm{lt}(m) = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$ for all $s, t \in \mathbb{F}^{\times}$, and $m \in \mathbb{F}$ and set $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Moreover, for matrices X and Y, with Y invertible, let $X^Y := Y^{-1}XY$ and ${}^{Y}X := YXY^{-1}$ denote the conjugations of X by Y.

In Section 3.1, we present Kubota's 2-cocycle for G [Kub67] and G' [Kub69], which leads to the construction of their n-fold covering groups $\widetilde{G}$ and $\widetilde{G'}$ respectively. In Section 3.2 and 3.3, we describe the structure of certain subgroups of $\widetilde{G}$ and $\widetilde{G'}$.

3.1 Construction

3.1.1 Covering Groups of $\mathrm{SL}_2(\mathbb{F})$

Define the map $\beta : G \times G \to \mu_n$ by

$$\beta(\mathbf{g}_1, \mathbf{g}_2) = \left(\frac{X(\mathbf{g}_1\mathbf{g}_2)}{X(\mathbf{g}_1)}, \frac{X(\mathbf{g}_1\mathbf{g}_2)}{X(\mathbf{g}_2)} \right)_n, \tag{3.1.1}$$

where $(\ ,\ )_n$ denotes the n-th Hilbert symbol, and

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{otherwise.} \end{cases}$$

Note that $X(\mathbf{g}) \neq 0$ for every $\mathbf{g} \in G$, so β is well-defined. The following proposition is the main result in Kubota's 1967 paper [Kub67].

Proposition 3.1.1. *The map β is a 2-cocycle that is not a 2-coboundary, and it determines an n-fold topological covering group of G.*

It is known that there exists a unique non-trivial n-fold cover of G [Moo68]. We denote the n-fold topological covering of G constructed via the 2-cocycle β by $\widetilde{G}$. Recall from Definition 2.1.5 that $\widetilde{G} = G \times \mu_n = \{(\mathbf{g}, \zeta) |\ \mathbf{g} \in G \text{ and } \zeta \in \mu_n\}$ as a set, and the multiplication in $\widetilde{G}$ is given by by

$$(\mathbf{g}_1, \zeta_1)(\mathbf{g}_2, \zeta_2) = (\mathbf{g}_1\mathbf{g}_2, \beta(\mathbf{g}_1, \mathbf{g}_2)\zeta_1\zeta_2),$$

for all $\mathbf{g}_1, \mathbf{g}_2 \in G$ and $\zeta_1, \zeta_2 \in \mu_n$. Moreover, recall that the n-fold covering group $\widetilde{G}$ fits into the exact sequence

$$0 \to \mu_n \xrightarrow{i} \widetilde{G} \xrightarrow{\mathsf{p}} G \to 0, \tag{3.1.2}$$

where the group homomorphisms i and p are defined to be $i(\zeta) = (1, \zeta)$ and $\mathsf{p}(\mathbf{g}, \zeta) = \mathbf{g}$ for all $\zeta \in \mu_n$ and $\mathbf{g} \in G$.

Lemma 3.1.2. *The identity element in $\widetilde{G}$ is $(\mathrm{I}_2, 1)$. Moreover, for $\mathbf{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $\zeta \in \mu_n$, the inverse of $(\mathbf{g}, \zeta) \in \widetilde{G}$ given by*

$$(\mathbf{g}, \zeta)^{-1} = \begin{cases} (\mathbf{g}^{-1}, \zeta^{-1}), & \text{if} \quad c \neq 0 \\ (\mathbf{g}^{-1}, (a,d)_n \zeta^{-1}), & \text{otherwise.} \end{cases}$$

Proof: The first statement follows from observing that for all $\mathbf{g} \in G$, $\beta(\mathrm{I}_2, \mathbf{g}) = \beta(\mathbf{g}, I_2) = 1$. Note that $\mathbf{g}^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. If $c \neq 0$ then we compute the 2-cocycle

$$\beta(\mathbf{g}, \mathbf{g}^{-1}) = \left(c^{-1}, -c^{-1}\right)_n = (c, -c)_n = 1.$$

So $(\mathbf{g}, \zeta)^{-1} = (\mathbf{g}^{-1}, \zeta^{-1})$. If $c = 0$, then

$$\beta(\mathbf{g}, \mathbf{g}^{-1}) = (d^{-1}, a^{-1})_n = (d, a)_n.$$

Hence, $(\mathbf{g}, \zeta)(\mathbf{g}^{-1}, (a,d)_n \zeta^{-1}) = (\mathrm{I}_2, (a,d)_n (d,a)_n \zeta \zeta^{-1}) = (\mathrm{I}_2, 1)$. ∎

3.1.2 Covering Groups of $\mathrm{GL}_2(\mathbb{F})$

In 1969, Kubota extended the map β to a 2-cocycle β' for G' in [Kub69]. Here, we describe the construction of β'. For each $x \in \mathbb{F}^\times$ and $\mathbf{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, define

$$v(x, \mathbf{g}) := \begin{cases} 1, & \text{if} \quad c \neq 0; \\ (x, d)_n, & \text{otherwise.} \end{cases}$$

Note that $v(1, \mathbf{g}) = 1$. Let us identify $\mathbb{F}^\times$ as a subgroup of G' by $\{\bar{y} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & y \end{smallmatrix}\right) \mid y \in \mathbb{F}^\times\}$. Then $\mathbb{F}^\times \ltimes G \cong G'$ via $(y, \mathbf{g}) \mapsto \bar{y}\mathbf{g}$. For each $y \in \mathbb{F}^\times$, define the map

$$\begin{aligned} \phi_y : \widetilde{G} &\to \widetilde{G} \\ (\mathbf{g}, \zeta) &\mapsto (\mathbf{g}^{\bar{y}}, v(y, \mathbf{g})\zeta). \end{aligned}$$

It is proved in [Kub69], using a case-by-case argument, that the map

$$\begin{aligned} \mathbb{F}^\times &\to \mathrm{Aut}(\widetilde{G}) \\ y &\mapsto \phi_y \end{aligned}$$

is well-defined and, indeed, a group homomorphism. Hence, it defines a semi-direct product $\mathbb{F}^\times \ltimes_\phi \widetilde{G}$, with multiplication given by

$$(y_1, (\mathbf{g}_1, \zeta_1))(y_2, (\mathbf{g}_2, \zeta_2)) = (y_1 y_2, (\mathbf{g}_1, \zeta_1)(\mathbf{g}_2^{\bar{y}_1}, v(y_1, \mathbf{g}_2)\zeta_2)), \tag{3.1.3}$$

for $y_1, y_2 \in \mathbb{F}^\times, \mathbf{g}_1, \mathbf{g}_2 \in G$ and $\zeta_1, \zeta_2 \in \mu_n$. The group $\mathbb{F}^\times \ltimes_\phi \widetilde{G}$ is an n-fold topological covering of G' via the map $(y, (\mathbf{g}, \zeta)) \mapsto \bar{y}\mathbf{g}$. The 2-cocycle β' associated to this covering arises in transforming the group structure of $\mathbb{F}^\times \ltimes_\phi \widetilde{G}$ to the set $G' \times \mu_n$ via the bijections

$$\begin{aligned} (y, (\mathbf{g}, \zeta)) &\mapsto (\bar{y}\mathbf{g}, \zeta) \\ (\det(\mathbf{g}'), (a(\mathbf{g}')\mathbf{g}', \zeta)) &\leftarrow\!\shortmid (\mathbf{g}', \zeta), \end{aligned} \tag{3.1.4}$$

where $a(\mathbf{g}') := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \det^{-1}(\mathbf{g}') \end{smallmatrix}\right)$, so $a(\mathbf{g}')\mathbf{g}' \in G$.

A straightforward calculation yields the map $\beta' : G' \times G' \to \mu_n$ given by

$$\beta'(\mathbf{g}_1, \mathbf{g}_2) = \beta\left({}^{a(\mathbf{g}_2)}\left(a(\mathbf{g}_1)\mathbf{g}_1\right), a(\mathbf{g}_2)\mathbf{g}_2\right) \, v\left(\det(\mathbf{g}_2), a(\mathbf{g}_1)\mathbf{g}_1\right), \tag{3.1.5}$$

where β is the Kubota 2-cocycle for G defined in (3.1.1). Note that β' restricted to $G \times G$ coincides with β. Denote this n-fold covering group of G' associated β' by $\widetilde{G'}$; so, $\widetilde{G'} = G' \times \mu_n$ as a set, and the multiplication is given by twisting with β'.

Kubota proved the following proposition directly in [Kub69].

Proposition 3.1.3. *The map β' is a 2-cocycle that is not a 2-coboundary, and it determines an n-fold covering group of G'.*

The covering group $\widetilde{G'}$ fits into the exact sequence

$$0 \to \mu_n \xrightarrow{i'} \widetilde{G'} \xrightarrow{\mathsf{p}'} G' \to 0, \tag{3.1.6}$$

where i' and p' are group homomorphisms defined by $i'(\zeta) = (1, \zeta)$ and $\rho'(\mathbf{g}, \zeta) = \mathbf{g}$ for all $\zeta \in \mu_n$ and $\mathbf{g} \in G'$.

Lemma 3.1.4. *The identity element in* $\widetilde{G'}$ *is* $(\mathrm{I}_2, 1)$. *Moreover, for* $\mathbf{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G'$ *and* $\zeta \in \mu_n$, *the inverse of* $(\mathbf{g}, \zeta) \in \widetilde{G'}$ *is*

$$(\mathbf{g}, \zeta)^{-1} = \begin{cases} (\mathbf{g}^{-1}, \zeta^{-1}) & \text{if} \quad c \neq 0 \\ \left(\mathbf{g}^{-1}, \left(a \ \det(\mathbf{g}), d \det(\mathbf{g})^{-1}\right)_n \zeta^{-1}\right) & \text{otherwise.} \end{cases}$$

Proof: It is not difficult to see that $\beta'(\mathrm{I}_2, \mathbf{g}) = \beta'(\mathbf{g}, \mathrm{I}_2) = 1$, which implies that $(\mathrm{I}_2, 1)$ is the identity element in $\widetilde{G'}$. Note $\mathbf{g}^{-1} = \det(\mathbf{g})^{-1} \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Let us calculate $\beta'(\mathbf{g}, \mathbf{g}^{-1})$ using the definition of β' in (3.1.5). Observe that

$$^{a(\mathbf{g}^{-1})}\left(a(\mathbf{g})\mathbf{g}\right) = \mathbf{g}a(\mathbf{g}) = \begin{pmatrix} a & b \det(\mathbf{g})^{-1} \\ c & d \det(\mathbf{g})^{-1} \end{pmatrix},$$

and

$$a(\mathbf{g}^{-1})\mathbf{g}^{-1} = \begin{pmatrix} d \det(\mathbf{g})^{-1} & -b \det(\mathbf{g})^{-1} \\ -c & a \end{pmatrix}.$$

Hence, by (3.1.1),

$$\beta\left(^{a(\mathbf{g}^{-1})}\left(a(\mathbf{g})\mathbf{g}\right), a(\mathbf{g}^{-1})\mathbf{g}^{-1}\right) = \begin{cases} (c, -c)_n = 1, & \text{if } c \neq 0 \\ (d \ \det(\mathbf{g})^{-1}, a)_n, & \text{otherwise.} \end{cases}$$

Moreover, observe that

$$a(\mathbf{g})\mathbf{g} = \begin{pmatrix} a & b \\ c \det(\mathbf{g})^{-1} & d \det(\mathbf{g})^{-1} \end{pmatrix}.$$

Therefore, $v\left(\det(\mathbf{g}^{-1}), a(\mathbf{g})\mathbf{g}\right)$ is 1 if $c \neq 1$, and is $\left(\det(\mathbf{g}^{-1}), d \det(\mathbf{g}^{-1})\right)_n$ otherwise. The result follows, as in the proof of Lemma 3.1.2, by noting that

$$\left(d \ \det(\mathbf{g})^{-1}, a\right)_n \left(\det(\mathbf{g}^{-1}), d \det(\mathbf{g}^{-1})\right)_n = \left(d \det(\mathbf{g})^{-1}, a \ \det(\mathbf{g})\right)_n.$$

∎

Remark 3.1.5. Note that because β' extends β, certain properties of β can be deduced from those of β'. For instance, Lemma 3.1.2 is a special case of Lemma 3.1.4.

3.2 Maximal Toral Subgroups

3.2.1 Tori of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$

Definition 3.2.1. *Let $T = \{\mathrm{dg}(t) \mid t \in \mathbb{F}^\times\}$ be the standard maximal torus in G. Define the* metaplectic torus *$\widetilde{T}$ of $\widetilde{G}$ to be the inverse image of T under the projection map p in (3.1.2).*

We will see shortly that $\widetilde{T}$ is not abelian, however, with an abuse of notation, we call it a torus for the covering group $\widetilde{G}$. The group multiplication of the metaplectic torus $\widetilde{T}$ can be simplified as follows.

Lemma 3.2.2. *Let $(\mathrm{dg}(t_1), \zeta_1)$ and $(\mathrm{dg}(t_2), \zeta_2)$ be in $\widetilde{T}$. Then*

$$(\mathrm{dg}(t_1), \zeta_1)\,(\mathrm{dg}(t_2), \zeta_2) = (\mathrm{dg}(t_1 t_2), (t_2, t_1)_n \zeta_1 \zeta_2)\,,$$

and

$$(\mathrm{dg}(t), \zeta)^{-1} = (\mathrm{dg}(t^{-1}), (t, t)_n \zeta^{-1}).$$

Proof: The key is to calculate the cocycle β (3.1.1) on toral elements:

$$\beta\,(\mathrm{dg}(t_1), \mathrm{dg}(t_2)) = \left(\frac{(t_1 t_2)^{-1}}{t_1^{-1}}, \frac{(t_1 t_2)^{-1}}{t_2^{-1}}\right)_n = \left(t_2^{-1}, t_1^{-1}\right)_n = (t_2, t_1)_n,$$

for all $t_1, t_2 \in T$. The inverse formula follows directly from that for $\widetilde{G}$ given in Lemma 3.1.2, and Proposition 2.2.1, which states that $(t, t^{-1})_n = (t, t)_n$. ∎

Lemma 3.2.3. *The group $\widetilde{T}$ is commutative if and only if $n = 2$.*

Proof: Lemma 3.2.2 implies that $\widetilde{T}$ is commutative if and only if $(t_2, t_1)_n = (t_1, t_2)_n$ for all $t_1, t_2 \in \mathbb{F}^\times$, which, by Proposition 2.2.1, is equivalent to $(t_2, t_1)_n^2 = 1$, for all $t_1, t_2 \in \mathbb{F}^\times$. Let ζ denote the primitive n-th root of unity, and choose t_1 and t_2 such that $(t_2, t_1)_n = \zeta$. Hence, $\zeta^2 = 1$, which implies $n = 2$. ∎

Corollary 3.2.4. *The central extension $\widetilde{G}$ does not split over T, for $n > 2$.*

Proof: If the central extension were to split over T then we would have $\widetilde{T} = T \times \mu_n$, which is an abelian group. ∎

Proposition 3.2.5. *The group $\widetilde{T}$ is a Heisenberg group.*

In order to prove Proposition 3.2.5, first we need to calculate the commutator subgroup $[\widetilde{T}, \widetilde{T}]$ of $\widetilde{T}$. For elements $(\mathrm{dg}(t_1), \zeta_1)$ and $(\mathrm{dg}(t_2), \zeta_2)$ in $\widetilde{T}$, their commutator is

$$[(\mathrm{dg}(t_1), \zeta_1), (\mathrm{dg}(t_2), \zeta_2)] = (\mathrm{dg}(t_1), \zeta_1)^{-1}(\mathrm{dg}(t_2), \zeta_2)^{-1}(\mathrm{dg}(t_1), \zeta_1)(\mathrm{dg}(t_2), \zeta_2).$$

Note that by Lemma 3.2.2,

$$(\mathrm{dg}(t_1), \zeta_1)^{-1}(\mathrm{dg}(t_2), \zeta_2)^{-1} = (\mathrm{dg}(t_1^{-1}t_2^{-1}), (t_2, t_1)_n(t_1, t_1)_n(t_2, t_2)_n\zeta_1\zeta_2).$$

So, $[(\mathrm{dg}(t_1), \zeta_1), (\mathrm{dg}(t_2), \zeta_2)]$ is equal to

$$\left(\mathrm{I}_2, (t_1t_2, t_1^{-1}t_2^{-1})_n(t_2, t_1)_n(t_1, t_1)_n(t_2, t_2)_n(t_2, t_1)_n\right),$$

which, using the properties of Hilbert symbol in Propositions 2.2.1 and 2.2.5, simplifies to $(\mathrm{I}_2, (t_2, t_1)_n^2)$.

Lemma 3.2.6. *The commutator subgroup $[\widetilde{T}, \widetilde{T}]$ is isomorphic to $\mu_{\underline{n}}$.*

Proof: It follows from the formula for the Hilbert symbol in Proposition 2.2.2 that, as t_2 and t_1 range over $\mathbb{F}^\times$, $(t_2, t_1)_n$ exhaust μ_n. Therefore,

$$[\widetilde{T}, \widetilde{T}] = \langle (1, \zeta^2) \mid \zeta \in \mu_n \rangle,$$

which is isomorphic to μ_n when n is odd and to $\mu_{\frac{n}{2}}$ when n is even. ∎

Proof: (**of Proposition 3.2.5**)
By Lemma 3.2.6, the commutator subgroup of $\widetilde{T}$ is central in $\widetilde{T}$. Hence, Lemma 1.2.3 implies that $\widetilde{T}$ is a Heisenberg group. ∎

We know that μ_n in central in $\widetilde{T}$. Next we calculate the centre of $\widetilde{T}$ precisely.

Lemma 3.2.7. *The centre of* $\widetilde{T}$ *is*

$$Z(\widetilde{T}) = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times \underline{n}}, \zeta \in \mu_n\}.$$

Proof: If $t \in \mathbb{F}^{\times \underline{n}}$, then $t^2 \in \mathbb{F}^{\times n}$ and hence by Proposition 2.2.1, $(t^2, t')_n = 1$ for all $t' \in \mathbb{F}^\times$. On the other hand, the same proposition implies that $(t^2, t')_n = (t, t')_n (t', t)_n^{-1}$. Hence, $(t, t')_n = (t', t)_n$ for all $t' \in \mathbb{F}^\times$, which by Lemma 3.2.2 implies $(\mathrm{dg}(t), \zeta) \in Z(\widetilde{T})$ for any $\zeta \in \mu_n$.

Conversely, let $(\mathrm{dg}(t), \zeta) \in Z(\widetilde{T})$. Then, by Lemma 3.2.2, $(t, t')_n = (t', t)_n$ for all $t' \in \mathbb{F}^\times$, as above we conclude that $(t, t')^2 = (t^2, t') = 1$ for all $t' \in \mathbb{F}^\times$. Proposition 2.2.1 then implies that $t^2 \in \mathbb{F}^{\times n}$, so, there exists an $x \in \mathbb{F}^\times$ such that $t^2 = x^n$.

If n is even, then $t^2 - x^n = t^2 - x^{2\underline{n}} = (t - x^{\underline{n}})(t + x^{\underline{n}}) = 0$. This implies $t = \pm x^{\underline{n}}$. Let ξ be a primitive n-th root of unity in $\mu_n \subset \mathbb{F}$. Then, $-1 = \xi^{\underline{n}}$ is an $\underline{n}$-th power and hence, in either case, $t \in \mathbb{F}^{\times \underline{n}}$.

Now suppose that $n = 2m+1$, for some integer m. Without loss of generality, we can assume that x and t are in $\mathcal{O}$ (since one can multiply both sides of the equation $t^2 = x^n$ by ϖ^{2nk} for a suitable choice of k). We will show that the polynomial $f(X) = X^n - t$ has a root in $\mathcal{O}$ and hence, $t \in \mathcal{O}^n$. Let $\bar{t}$ and $\bar{x}$ denote the images of

t and x in the residue field. Thus we have

$$1 = \bar{t}^{(q-1)} = \bar{t}^{2(\frac{q-1}{2})} = \bar{x}^{n(\frac{q-1}{2})} = \bar{x}^{(\frac{q-1}{2})2m}\bar{x}^{(\frac{q-1}{2})} = \bar{x}^{\frac{q-1}{2}}.$$

So, because $\kappa^\times$ is a cyclic group, there exists $\bar{y} \in \kappa^\times$ such that $\bar{x} = \bar{y}^2$. So we have,

$$\bar{t}^2 - \bar{x}^n = \bar{t}^2 - \bar{y}^{2n} = (\bar{t} - \bar{y}^n)(\bar{t} + \bar{y}^n) = 0.$$

Hence, $\bar{t} = \bar{y}^n$ or $\bar{t} = -\bar{y}^n = (-\bar{y})^n$. That means the polynomial $f(X)$ has a root $\alpha = -\bar{y}$ or $\alpha = \bar{y}$ in κ. Note that $f'(\alpha) = n\alpha \neq 0$ in κ, since $(n, p) = 1$. By Hensel's Lemma 2.2.10, $f(X)$ admits a solution in $\mathcal{O}$. Hence, t is an n-th power in $\mathcal{O}$. ∎

Lemma 3.2.8. *The index of $Z(\widetilde{T})$ in $\widetilde{T}$ is $\underline{n}^2$.*

Proof: It is clear from Lemma 3.2.7 that $[\widetilde{T} : Z(\widetilde{T})] = [\mathbb{F}^\times : \mathbb{F}^{\times \underline{n}}]$. Because $\mathbb{F}^\times \cong \mathcal{O}^\times \times \mathbb{Z}$, $[\mathbb{F}^\times : \mathbb{F}^{\times \underline{n}}] = [\mathcal{O}^\times \times \mathbb{Z} : \mathcal{O}^{\times \underline{n}} \times \underline{n}\mathbb{Z}] = \underline{n}[\mathcal{O}^\times : \mathcal{O}^{\times \underline{n}}]$, which is equal to $\underline{n}^2$ by Lemma 2.2.12. ∎

Remark 3.2.9. Set $T^n := \{\mathrm{dg}(t) \mid t \in \mathbb{F}^{\times n}\}$, and let $\widetilde{T^n} = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times n}, \zeta \in \mu_n\} \subset \widetilde{T}$. This is equal to $Z(\widetilde{T})$ if n is odd, but is of index 4 in $Z(\widetilde{T})$ if n is even. It follows directly from Lemma 3.2.2 and Proposition 2.2.1 that the central extension $\widetilde{G}$ splits trivially over T^n. That is, $\widetilde{T^n} \cong T^n \times \mu_n$.

3.2.2 Tori of $\widetilde{\mathrm{GL}}_2(\mathbb{F})$

Definition 3.2.10. *Let $T' = \{\mathrm{dg}(s, t) \mid s, t \in \mathbb{F}^\times\}$ be the standard maximal torus in G'. Define the metaplectic torus $\widetilde{T'}$ of $\widetilde{G'}$ to be the inverse image of T' under p' in* (3.1.6).

The following lemma gives the simplified formulas for calculation in $\widetilde{T'}$.

Lemma 3.2.11. *Let* $\mathbf{g}_1 = \mathrm{dg}(s_1, t_1)$ *and* $\mathbf{g}_2 = \mathrm{dg}(s_2, t_2)$ *be in* T'. *Then*

$$(\mathbf{g}_1, \zeta_1)(\mathbf{g}_2, \zeta_2) = (\mathbf{g}_1\mathbf{g}_2, (s_1, t_2)_n \zeta_1 \zeta_2)$$

and

$$(\mathbf{g}_1, \zeta_1)^{-1} = (\mathbf{g}_1^{-1}, (s_1, t_1)_n \zeta^{-1}).$$

Proof: We calculate β' by applying (3.1.5) to $\mathbf{g}_1$ and $\mathbf{g}_2$. Note that ${}^{a(\mathbf{g}_2)}(a(\mathbf{g}_1)\mathbf{g}_1) = \mathrm{dg}(s_1, s_1^{-1})$, $a(\mathbf{g}_2)\mathbf{g}_2 = \mathrm{dg}(s_2, s_2^{-1})$, and $v(\det(g_2), a(g_1)g_1) = (s_2t_2, s_1^{-1})_n$. Hence, $\beta'(g_1, g_2) = (s_2, s_1)_n(s_2t_2, s_1^{-1}) = (s_2, s_1)_n(s_2, s_1^{-1})_n(t_2, s_1^{-1})_n = (s_1, t_2)_n$. The second statement is a direct result of Lemma 3.1.4. ∎

Remark 3.2.12. Note that it follows that $\widetilde{T'}$ is not abelian, and hence the central extension $\widetilde{G'}$ does not split over T'.

Proposition 3.2.13. *The commutator subgroup* $[\widetilde{T'}, \widetilde{T'}]$ *is the central subgroup* μ_n, *and so* $\widetilde{T'}$ *is a Heisenberg group.*

Proof: Let $g_1 = \mathrm{dg}(s_1, t_1)$ and $g_2 = \mathrm{dg}(s_2, t_2)$. Then

$$\begin{aligned}[(g_1, \zeta_1), (g_2, \zeta_2)] &= ({g_1}^{-1}, (s_1, t_1)_n {\zeta_1}^{-1})({g_2}^{-1}, (s_2, t_2)_n {\zeta_2}^{-1})(g_1, \zeta_1)(g_2, \zeta_2) \\ &= (\mathrm{I}_2, (s_1, t_2)_n^2 (s_1, t_1)_n (s_2, t_2)_n).\end{aligned}$$

Suppose $s_1 = 1$. Then $(s_1, t_2)_n^2 (s_1, t_1)_n (s_2, t_2)_n = (s_2, t_2)_n$ and $(s_2, t_2)_n$ exhaust μ_n as s_2 and t_2 range over $\mathbb{F}^\times$. Hence, $[\widetilde{T'}, \widetilde{T'}] = \{(\mathrm{I}_2, \zeta) \mid \zeta \in \mu_n\} = i(\mu_n)$. Hence, $[\widetilde{T'}, \widetilde{T'}]$ is central in $\widetilde{G'}$ and because $Z(\widetilde{G'}) \subseteq Z(\widetilde{T'})$, Lemma 1.2.3 implies that $\widetilde{T'}$ is a Heisenberg group. ∎

Lemma 3.2.14. *The centre of* $\widetilde{T'}$ *is*

$$Z(\widetilde{T'}) = \{(\mathrm{dg}(s, t), \zeta) \mid s, t \in {\mathbb{F}^\times}^n, \zeta \in \mu_n\}.$$

Proof: It follows from Lemma 3.2.11 that $(\mathrm{dg}(s_1, t_1), \zeta_1) \in Z(\widetilde{T'})$ if and only if $(s_1, t)_n(t_1, s)_n = 1$ for all $s, t \in \mathbb{F}^\times$, which implies that s_1 and t_1 are in $\mathbb{F}^{\times^n}$. ∎

Lemma 3.2.15. *The index of $Z(\widetilde{T'})$ in $\widetilde{T'}$ is n^4.*

Proof: Note that $[\widetilde{T'} : Z(\widetilde{T'})] = [\mathbb{F}^\times : \mathbb{F}^{\times^n}][\mathbb{F}^\times : \mathbb{F}^{\times^n}]$. The rest of the argument is similar to the proof of Lemma 3.2.8. ∎

Set $T'^n = \{\mathrm{dg}(s, t) \mid s, t \in \mathbb{F}^{\times^n}\}$, so $Z(\widetilde{T'}) = \mathrm{p}'^{-1}(T'^n)$. It follows from Lemma 2.2.7 that the central extension $\widetilde{G'}$ splits trivially over T'^n.

3.3 Structure Theory

3.3.1 Some Subgroups of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$

Our first subgroups of interest are coverings of the standard Borel subgroup in G.

Definition 3.3.1. *Let $B := \{\left(\begin{smallmatrix} t & m \\ 0 & t^{-1}\end{smallmatrix}\right) \mid t \in \mathbb{F}^\times, m \in \mathbb{F}\}$ denote the standard Borel subgroup of G, and $N = \{\mathrm{ut}(m) \mid m \in \mathbb{F}\}$ be its unipotent radical. Define the* metaplectic Borel $\widetilde{B} = \mathsf{p}^{-1}(B)$, *where p is the projection map in* (3.1.2). *Also, let $\widetilde{N} = \mathsf{p}^{-1}(N)$. So, $\widetilde{B} = \{(\mathbf{b}, \zeta) \mid \mathbf{b} \in B, \zeta \in \mu_n\}$ and $\widetilde{N} = \{(\mathrm{ut}(m), \zeta) \mid m \in \mathbb{F}, \zeta \in \mu_n\}$.*

Let $\mathbf{b}_1 = \begin{pmatrix} t_1 & m_1 \\ 0 & t_1^{-1}\end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} t_2 & m_2 \\ 0 & t_2^{-1}\end{pmatrix}$ for $t_1, t_2 \in \mathbb{F}^\times$, and $m_1, m_2 \in \mathbb{F}$. It is not difficult to see that

$$(\mathbf{b}_1, \zeta_1)(\mathbf{b}_2, \zeta_2) = (\mathbf{b}_1\mathbf{b}_2, (t_2, t_1)_n\zeta_1\zeta_2).$$

Lemma 3.3.2. *The central extension splits $\widetilde{G}$ trivially over N.*

Proof: From (3.1.1), it is easy to see that $\beta|_{N\times N} = 1$, and hence the central extension splits trivially over N. That is $\widetilde{N} \cong N \times \mu_n$ via the identity map. ∎

We can therefore identify N as a subgroup of $\widetilde{N}$ as follows:

$$N \cong (\mathrm{ut}(m), 1) \mid m \in \mathbb{F} \subset \widetilde{N}$$

Lemma 3.3.3. *The subgroup $\widetilde{B}$ is the semidirect product of $\widetilde{T}$ and N.*

Proof: First let us show that N (seen as a subgroup of $\widetilde{N}$) is invariant under conjugation by $\widetilde{T}$. Let $(\mathrm{dg}(t), \zeta) \in \widetilde{T}$ and $(\mathrm{ut}(m), 1) \in N$.

$$\begin{aligned} \left(\mathrm{dg}(t^{-1}), (t,t)\zeta^{-1}\right)(\mathrm{ut}(m), 1)(\mathrm{dg}(t), \zeta) &= \left(\mathrm{ut}(mt^{-2}), (t,t)(t,t^{-1})\zeta^{-1}\zeta\right) \\ &= \left(\mathrm{ut}(mt^{-2}), 1\right) \in N. \end{aligned}$$

Moreover, the Levi decomposition $B = TN$ in G implies

$$\left(\begin{pmatrix} t & m \\ 0 & t^{-1} \end{pmatrix}, \zeta\right) = (\mathrm{dg}(t), \zeta)\left(\mathrm{ut}(mt^{-1}), 1\right),$$

for all $t \in \mathbb{F}^\times$ and $m \in \mathbb{F}$, and hence $\widetilde{B} = \widetilde{T}N$. The result follows from observing that $\widetilde{T} \cap N = \{\mathrm{I}_2\}$. ∎

Next, we describe a family of compact open subgroups of $\widetilde{G}$. Let $K = SL_2(\mathcal{O})$ be a maximal compact subgroup of G and let $\widetilde{K}$ denote the inverse image of K under p. The following proposition is proven in [Kub69]. [1]

Proposition 3.3.4. *The central extension $\widetilde{G}$ splits over K. Moreover, the splitting function $s : K \to \mu_n$ is*

$$s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} (c,d)_n & 0 < \mathrm{val}(c) < \infty \\ 1 & \textit{otherwise.} \end{cases} \tag{3.3.1}$$

[1]Proposition 3.3.4 only holds when $(n,p) = 1$, which is implied by our assumption of $n | q-1$.

In other words, Proposition 3.3.4 states that $\widetilde{K} \cong K \times \mu_n$, where the isomorphism is given by

$$\begin{aligned} \widetilde{K} &\to K \times \mu_n \\ (\mathbf{k}, \zeta) &\mapsto (\mathbf{k}, s(\mathbf{k})\zeta). \end{aligned} \tag{3.3.2}$$

The inverse image of K in $\widetilde{K}$ under the isomorphism (3.3.2) is the subgroup

$$\widetilde{K}_0 := \{(\mathbf{k}, s(\mathbf{k})^{-1}) \mid \mathbf{k} \in K\}$$

of $\widetilde{K}$. Recall from Example 1.1.4 that the congruent subgroups

$$K_j := \{\mathbf{g} \in K \mid \mathbf{g} \equiv \mathrm{I}_2 \mod \mathfrak{p}^j\}, \quad j \geq 1,$$

are compact open in G, and give a fundamental system of open neighbourhoods of I_2.

Lemma 3.3.5. *The central extension $\widetilde{K}$ splits trivially over each of the subgroups K_j, $j \geq 1$, $T \cap K$ and $B \cap K$.*

Proof: Let $\mathbf{k} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_j$. Since $d \in 1 + \mathfrak{p}$, $s(\mathbf{k}) = (c, d)_n = 1$ by Lemma 2.2.7. On the other hand, it follows directly from (3.3.1) that if $\mathbf{k} \in B \cap K$ (or $T \cap K$) then $s(\mathbf{k}) = 1$. Thus, in all these cases, the splitting is trivial. ∎

Remark 3.3.6. We identify $K_j \cong K_j \times \{1\}$, $j \geq 1$, $B \cap K \cong (B \cap K) \times \{1\}$ and $T \cap K \cong (T \cap K) \times \{1\}$ as subgroups of $\widetilde{K}$.

Remark 3.3.7. The open compact subgroups K_j, $j \geq 1$ give a fundamental system of open neighbourhoods of the identity. Hence, $\widetilde{G}$ is a locally profinite group.

Let $\widetilde{I} = \{\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta\right) \in \widetilde{K} \mid c \in \mathfrak{p}, a, b, d \in \mathcal{O}\}$ be the inverse image of Iwahori subgroup of G in $\widetilde{G}$. The next lemma verifies that the well-known decompositions of G hold for $\widetilde{G}$.

Lemma 3.3.8. *Let* $N, \widetilde{T}, \widetilde{B}, \widetilde{K}_0$ *and* $\widetilde{I}$ *be the subgroups of* $\widetilde{G'}$ *as defined above, and* $\widetilde{w} := (w, 1)$. *Then the following decompositions hold.*

(i) $\widetilde{G} = N\widetilde{T}\widetilde{K}_0$, *the Iwasawa decomposition,*

(ii) $\widetilde{K} = \widetilde{I}\widetilde{w}\widetilde{I} \cup \widetilde{I}$,

(iii) $\widetilde{G} = \widetilde{B}\widetilde{w}\widetilde{B} \cup \widetilde{B}$, *the Bruhat decomposition,*

(iv) $\widetilde{K} = (\widetilde{B} \cap \widetilde{K})\widetilde{w}(N \cap \widetilde{K}) \cup \widetilde{I}$.

Proof: Let $(\mathbf{g}, \zeta) \in \widetilde{G}$. By the Iwasawa decomposition for G, $\mathbf{g} = \mathbf{ntk}$ for some $\mathbf{n} \in N, \mathbf{t} \in T$ and $\mathbf{k} \in K$. We can write

$$(\mathbf{g}, \zeta) = \big(\mathbf{n}, 1\big)\big(\mathbf{t}, \zeta s(\mathbf{k})\beta^{-1}(\mathbf{nt}, \mathbf{k})\big)\big(\mathbf{k}, s(\mathbf{k})^{-1}\big),$$

where s is the splitting map in Proposition 3.3.4. This proves the Iwasawa decomposition for $\widetilde{G}$. The rest can be seen via the decomposition

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta) = (\mathrm{ut}(ac^{-1}), 1)(\mathrm{dg}(-c^{-1}), \zeta)\widetilde{w}(\mathrm{ut}(dc^{-1}), 1).$$

for $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta)$ in $\widetilde{G}$, $c \neq 0$. ∎

We normalize the Haar measure $\mu_{\widetilde{G}}$ of $\widetilde{G}$ so that $\mu_{\widetilde{G}}(\widetilde{K}) = 1$. Under this assumption, using the decompositions in Lemma 3.3.8, we compute the measure of $\widetilde{I}$ and $(\widetilde{B} \cap \widetilde{K})\widetilde{w}(N \cap \widetilde{K})$, which will be used in Chapter 5.

Lemma 3.3.9. *If* $\mu_{\widetilde{G}}(\widetilde{K}) = 1$ *then* $\mu_{\widetilde{G}}(\widetilde{I}) = \frac{1}{q+1}$ *and* $\mu_{\widetilde{G}}\Big((\widetilde{B} \cap \widetilde{K})\widetilde{w}(N \cap \widetilde{K})\Big) = \frac{q}{q+1}$, *where* q *is the size of the residue field.*

Proof: Let us first calculate the measure of the Iwahori subgroup $\widetilde{I}$. Let p be the canonical projection of $\widetilde{K}$ onto $\mathrm{SL}_2(\mathcal{O}/\mathfrak{p})$. Then $\widetilde{I}$ is the inverse image of $\mathcal{I} := \{\big(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right), \zeta\big) |\ a \in \kappa^\times,\ b \in \kappa,\ \zeta \in \mu_n\}$. Therefore, $[\mathrm{SL}_2(\mathcal{O}/\mathfrak{p}) : \mathcal{I}] = [\widetilde{K} : \widetilde{I}]$.

Therefore, $\mu(\widetilde{I}) = [\widetilde{K} : \widetilde{I}]^{-1} = [SL_2(\mathcal{O}/\mathfrak{p}) : \mathcal{I}]^{-1}$. By elementary counting, we get $|\mathcal{I}| = q(q-1)$ and $|\mathrm{SL}_2(\mathcal{O}/\mathfrak{p})| = q(q-1)(q+1)$. So we have $\mu_{\widetilde{G}}(\widetilde{I}) = \frac{1}{q+1}$. It follows from Lemma 3.3.8 that $\mu_{\widetilde{G}}(\widetilde{B} \cap \widetilde{K})\widetilde{w}(N \cap \widetilde{K}) = \frac{q}{q+1}$. ∎

In Chapter 4, we will apply the Stone-von Neumann Theorem 1.2.2 to the Heisenberg group $\widetilde{T}$ in order to investigate its irreducible representations. To do so, we need to identify a maximal abelian subgroup A of the Heisenberg group $\widetilde{T}$. Lemma 3.3.5 implies that $\widetilde{T} \cap \widetilde{K}$ is abelian; however, it does not contain $Z(\widetilde{T})$, whence is not maximal. Nonetheless, if the centralizer of $\widetilde{T} \cap \widetilde{K}$ in $\widetilde{T}$, $C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$, is abelian, it is a maximal abelian subgroup (see below) and we could choose A to be $C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$.

Lemma 3.3.10. *The centralizer* $C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$ *is* $\{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^\times,\ \underline{n} | \mathrm{val}(a),\ \zeta \in \mu_n\}$. *Moreover,* $[\widetilde{T} : C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})] = \underline{n}$.

Proof: If $(\mathrm{dg}(a), \zeta) \in \widetilde{T}$ commutes with every $(\mathrm{dg}(t), \zeta') \in \widetilde{T} \cap \widetilde{K}$, we have $(t,a)_n = (a,t)_n$ and hence $(t,a)_n^2 = 1$ for all $t \in \mathcal{O}^\times$. Applying the formula in Definition 2.2.2

$$1 = (t,a)_n^2 = \overline{t^{\mathrm{val}(a)}}^{2(\frac{q-1}{n})}.$$

Therefore, it must be the case that for odd n, $n \mid \mathrm{val}(a)$ and for even n, $\frac{n}{2} \mid \mathrm{val}(a)$. The argument is reversible.

Moreover, for every $(\mathrm{dg}(t), \zeta) \in \widetilde{T}$, $t = u\varpi^{\mathrm{val}(t)}$, $u \in \mathcal{O}^\times$, and $\mathrm{val}(t) = \underline{n}r + i$, where $0 \le i < \underline{n}$. It is easy to verify that

$$(\mathrm{dg}(t), \zeta) = \left(\mathrm{dg}(u\varpi^{r\underline{n}}), (u\varpi^{r\underline{n}}, \varpi^i)_n \zeta\right)\left(\mathrm{dg}(\varpi^i), 1\right).$$

Hence, $\{(\mathrm{dg}(\varpi^i), 1) \mid 0 \le i < \underline{n}\}$ is a complete set of coset representatives for $C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$ in $\widetilde{T}$ and hence, $[\widetilde{T} : C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})] = \underline{n}$. ∎

Let $\mathsf{a}_1 = (\mathrm{dg}(u_1\varpi^{\underline{n}r_1}), \zeta_1)$ and $\mathsf{a}_2 = (\mathrm{dg}(u_2\varpi^{\underline{n}r_2}), \zeta_2)$ denote two typical elements in $C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$, where $u_1, u_2 \in \mathcal{O}^\times$, $r_1, r_2 \in \mathbb{Z}$ and $\zeta_1, \zeta_2 \in \mu_n$.

Lemma 3.3.11. *The subgroup* $C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$ *is a maximal abelian subgroup of* $\widetilde{T}$.

Proof: By the multiplication formula in Lemma 3.2.2, a_1 and a_2 in $C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$ commute if $(u_2\varpi^{\underline{n}r_2}, u_1\varpi^{\underline{n}r_1})_n^2 = 1$. Using the properties of the Hilbert symbol, we have

$$(u_2\varpi^{\underline{n}r_2}, u_1\varpi^{\underline{n}r_1})_n^2 = (u_2,\varpi^{r_1})_n^{2\underline{n}}(\varpi^{r_2},u_1)_n^{2\underline{n}}(\varpi^{r_2},\varpi^{\underline{n}r_1})_n^{2\underline{n}} = 1.$$

Hence, $C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$ is abelian. To see that it is maximal, note that $\widetilde{T}\cap\widetilde{K}\subset C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$. Hence, if $x\in\widetilde{T}$ commutes with every element in $C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$, it commutes with $\widetilde{T}\cap\widetilde{K}$ and hence is in $C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$. ∎

Set $A := C_{\widetilde{T}}(\widetilde{T}\cap\widetilde{K})$. It is not hard to see from the multiplication formula in Lemma 3.2.2 that $\beta\left(\mathrm{dg}(u_1\varpi^{\underline{n}r_1}), \mathrm{dg}(u_2\varpi^{\underline{n}r_2})\right) = 1$ when $\underline{n} = n$. However, when $\underline{n} = \frac{n}{2}$, $\beta\left(\mathrm{dg}(u_1\varpi^{\underline{n}r_1}), \mathrm{dg}(u_2\varpi^{\underline{n}r_2})\right) = \pm 1$ depending on the the parity of $\frac{n}{2}$. Hence, the central extension splits trivially over A for odd n. However, that is not always true for even n.

The next lemma gives us a better picture of the structure of A and $Z(\widetilde{T})$ when n is even. Set

$$S := \{\left(\mathrm{dg}(\varpi^{\underline{n}k}),\zeta\right) \mid k\in\mathbb{Z}\}.$$

Lemma 3.3.12. *Suppose* n *is even. Then* $Z(\widetilde{T})\cong\mathcal{O}^{\times\underline{n}}\times S$ *and* $A\cong\mathcal{O}^\times\times S$.

Proof: Let us identify $\mathcal{O}^\times$ with the subgroup $\{(\mathrm{dg}(a),1)\mid a\in\mathcal{O}^\times\}$ of A. It is not difficult to see that S and $\mathcal{O}$ are (normal) subgroups of A. Let $\left(\mathrm{dg}(u\varpi^{\underline{n}k}),\zeta\right)\in A$, where $u\in\mathcal{O}^\times$. Observe that

$$\left(\mathrm{dg}(u\varpi^{\underline{n}k}),\zeta\right) = (\mathrm{dg}(u),1)\left(\mathrm{dg}(\varpi^{\underline{n}k}),(u,\varpi^{\underline{n}k})\zeta\right).$$

Hence, $A=\mathcal{O}^\times S$. Moreover, $\mathcal{O}^\times\cap S=\{(\mathrm{I}_2,1)\}$. Similarly, $\mathcal{O}^{\times\underline{n}}\cong\{(\mathrm{dg}(a^{\underline{n}}),1)\mid a\in\mathcal{O}^\times\}$ and $Z(\widetilde{T})=\mathcal{O}^{\times\underline{n}}S$ and $\mathcal{O}^{\times\underline{n}}\cap S=\{(\mathrm{I}_2,1)\}$. Hence, under these identifications, one can see that $Z(\widetilde{T})\cong\mathcal{O}^{\times\underline{n}}\times S$ and $A\cong\mathcal{O}^\times\times S$. ∎

Figure 3.1 gives an inclusion diagram of the subgroups of A.

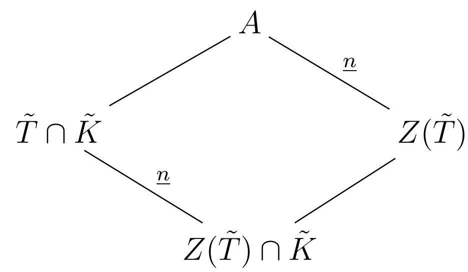


Figure 3.1: Inclusion diagram of the subgroups of $\widetilde{T}$, the symbols on the lines are the indices of subgroups: $[A : Z(\widetilde{T})] = [\widetilde{T} \cap \widetilde{K} : Z(\widetilde{T} \cap \widetilde{K})] = \underline{n}$.

Lemma 3.3.13. *The quotient $A/Z(\widetilde{T})$ is isomorphic to $\widetilde{T} \cap \widetilde{K}/(Z(\widetilde{T}) \cap \widetilde{K})$.*

Proof: Observe that $A = Z(\widetilde{T})(\widetilde{T} \cap \widetilde{K})$ and $Z(\widetilde{T}) \cap \widetilde{K} = \left(\widetilde{T} \cap \widetilde{K}\right) \cap Z(\widetilde{T})$. Hence, the result is a direct consequence of the second isomorphism theorem. ∎

3.3.2 Some Subgroups of $\widetilde{\mathrm{GL}}_2(\mathbb{F})$

The standard Borel subgroup $\widetilde{B'}$ of $\widetilde{G'}$ and its unipotent radical $\widetilde{N'}$ are defined analogously to those for $\widetilde{G}$. One can show similarly that the central extension splits trivially over N' and hence N' can be identified with a subgroup of $\widetilde{N'}$. It is not difficult to see that $\widetilde{B'}$ is semi-direct product of $\widetilde{T'}$ and N'.

Let $\widetilde{K'}$ denote the inverse image of the maximal compact subgroup $K' = \mathrm{GL}_2(\mathcal{O})$ of G'. It is proved by Kubota [Kub69] that the central extension of G' splits over K' via the splitting map [2]

$$s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} (c, d\ \det(\mathbf{g}))_n & \text{if } 0 < \mathrm{val}(c) < \infty \\ 1 & \text{otherwise.} \end{cases} \tag{3.3.3}$$

It follows from this formula that the central extension splits trivially over $B' \cap K'$ and $T' \cap K'$. Moreover, as in Lemma 3.3.5, one can identify the congruence subgroups K'_j,

[2]This result only holds when $(n, p) = 1$, which is implied by our assumption of $n | q - 1$.

$j \geq 1$, of K' as subgroups of $\widetilde{K'}$. Set $A' := C_{\widetilde{T'}}(\widetilde{T'} \cap \widetilde{K'})$, the centralizer of $\widetilde{T'} \cap \widetilde{K'}$ in $\widetilde{T'}$.

Proposition 3.3.14. *The group A' is a maximal abelian subgroup of $\widetilde{T'}$. Furthermore, $A' = \{(\mathrm{dg}(s,t),\zeta) \mid s,t \in \mathbb{F}^\times,\ n|\mathrm{val}(s), n|\mathrm{val}(t)\}$, and $[\widetilde{T'} : A'] = n^2$.*

Proof: Let $(\mathrm{dg}(s_1,t_1),\zeta_1)$ be in A'. Then $(\mathrm{dg}(s_1,t_1),\zeta_1)$ commutes with all $(\mathrm{dg}(s,t),\zeta)$ in $\widetilde{T'} \cap \widetilde{K'}$. Therefore, for all $s,t \in \mathcal{O}^\times$, $(s_1,t)_n(t_1,s)_n = 1$. That is,

$$\overline{t^{\mathrm{val}(s_1)}}^{\frac{q-1}{n}}\overline{s^{\mathrm{val}(t_1)}}^{\frac{q-1}{n}} = 1,$$

for all $s,t \in \mathcal{O}^\times$, which implies that $\mathrm{val}(s_1)$ and $\mathrm{val}(t_1)$ are both divisible by n. The reverse argument is trivial. It follows directly from the multiplication formula that A' is abelian and by an argument similar to the one in the proof of Lemma 3.3.11, A is a maximal abelian subgroup of $\widetilde{T'}$.

Moreover, similar to Lemma 3.3.10, one can see that $\{(\mathrm{dg}(\varpi^i,\varpi^j),1) \mid 0 \leq i,j < n\}$ is a complete set of coset representatives for A' in $\widetilde{T'}$ and hence, $[\widetilde{T'} : A'] = n^2$. ∎

It is not difficult to see that, regardless of the parity of n, A' splits trivially. Also, as with $\widetilde{T}$, $A'/Z(\widetilde{T'}) \cong \widetilde{T'} \cap \widetilde{K'}/(Z(\widetilde{T'}) \cap \widetilde{K'})$.

Chapter 4

K-Types of Principal Series Representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ and $\widetilde{\mathrm{GL}}_2(\mathbb{F})$

Recall that $G = \mathrm{SL}_2(\mathbb{F})$ and $G' = \mathrm{GL}_2(\mathbb{F})$. In this chapter, we first present the construction of the principal series representations of $\widetilde{G}$ and $\widetilde{G'}$ following [McN12]. We then go on to study the K-*types* of these representations.

The restriction of a smooth representation of $\widetilde{G}$ and $\widetilde{G'}$ to a maximal compact subgroup decomposes as a direct sum of irreducible representations (Proposition 1.1.28). The study of this decomposition, which is called the K-type problem, is motivated by a similar problem in the case of irreducible representations of real reductive groups. In the case of the linear p-adic groups, this problem has been solved for the principal series representations of GL(3) [CN09, CN10, OS14], and SL(2) [Nev05, Nev11], representations of GL(2) [Cas73] and supercuspidal representations of SL(2) [Nev13]. In this chapter, we solve the K-type problem for the principal series representations of $\widetilde{G}$ and $\widetilde{G'}$. Our main result is Theorem 4.4.19, which gives a complete decomposition of the restriction to $\widetilde{K}$ of the principal series representations of $\widetilde{G}$ into irreducible constituents, together with their multiplicities.

We continue assuming that $n|q-1$. Set $\underline{n} = n$ if n is odd and $\underline{n} = \frac{n}{2}$ if n is even. Moreover, set $\iota(t) = (\mathrm{dg}(t), 1)$ for all $t \in \mathbb{F}^\times$. We denote the elements of the linear group G by boldface font style, for example $\mathbf{g} \in G$; elements of the covering group $\widetilde{G}$ by typewriter font style, for example $\mathtt{g} \in \widetilde{G}$; and those of the p-adic field $\mathbb{F}$ in roman font style, for example $t \in \mathbb{F}$.

4.1 Principal Series Representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ and $\widetilde{\mathrm{GL}}_2(\mathbb{F})$

In this section, we apply the Stone-von Neumann Theorem 1.2.2 in order to construct irreducible representations of $\widetilde{T}$ and $\widetilde{T'}$. These are used to construct principal series representations of $\widetilde{G}$ and $\widetilde{G'}$ respectively. We are only interested in those representations of $\widetilde{G}$ ($\widetilde{G'}$) that do not factor through representations of G (G'), which we call *genuine* representations. To that end, let us fix a faithful character $\epsilon : \mu_n \to \mathbb{C}$. We consider only those representations of $\widetilde{G}$ ($\widetilde{G'}$) where the central subgroup μ_n acts by ϵ. In particular, a character of $Z(\widetilde{T})$ ($Z(\widetilde{T'})$) is called *genuine* if its subgroup μ_n acts by ϵ. Recall that $A = C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$ and $A' = C_{\widetilde{T'}}(\widetilde{T'} \cap \widetilde{K'})$ are maximal abelian subgroups of $\widetilde{T}$ and $\widetilde{T'}$ respectively.

Proposition 4.1.1. *Genuine irreducible smooth representations ρ and ρ' of $\widetilde{T}$ and $\widetilde{T'}$ are classified by genuine smooth characters of $Z(\widetilde{T})$ and $Z(\widetilde{T'})$ respectively. Moreover,* $\dim(\rho) = \underline{n}$ *and* $\dim(\rho') = n^2$.

Proof: Let χ and χ' denote genuine characters of $Z(\widetilde{T})$ and $Z(\widetilde{T'})$ respectively. Recall that by Propositions 3.2.5 and 3.2.13, $\widetilde{T}$ and $\widetilde{T'}$ are Heisenberg groups. By Lemma 3.2.6 and Proposition 3.2.13, the commutator subgroups $[\widetilde{T}, \widetilde{T}]$ and $[\widetilde{T'}, \widetilde{T'}]$ are equal to μ_n, where χ and χ' act by a faithful character ϵ. Thus, $\ker(\chi) \cap [\widetilde{T}, \widetilde{T}]$ and $\ker(\chi') \cap [\widetilde{T'}, \widetilde{T'}]$ are both trivial. So, the Stone-von Neumann theorem applies.

Let χ_0 and χ_0' be any extensions of χ and χ' to A and A' respectively. We apply the Stone-von Neumann Theorem 1.2.2 to construct unique smooth genuine representations $(\rho, \mathrm{Ind}_A^{\widetilde{T}}\chi_0)$ and $(\rho', \mathrm{Ind}_{A'}^{\widetilde{T'}}\chi_0')$ of $\widetilde{T}$ and $\widetilde{T'}$ with central characters χ and χ' respectively. Note that a basis for the space $\mathrm{Ind}_A^{\widetilde{T}}\chi_0$ $(\mathrm{Ind}_{A'}^{\widetilde{T'}}\chi_0')$ consists of functions with support on distinct cosets of A in $\widetilde{T}$ (A' in $\widetilde{T'}$). By Lemma 3.3.10, and Proposition 3.3.14, $[\widetilde{T} : A] = \underline{n}$, and $[\widetilde{T'} : A'] = n^2$. Hence, $\dim(\rho) = \underline{n}$ and $\dim(\rho') = n^2$. ∎

The *genuine principal series representations* of $\widetilde{G}$ are defined in [McN12] as follows. Let ρ be a genuine smooth irreducible representation of $\widetilde{T}$ extended trivially over N to a representation of $\widetilde{B} = \widetilde{T} \ltimes N$. The genuine principal series representation of $\widetilde{G}$ associated to ρ is defined to be $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$. These principal series representations are not always irreducible; we investigate this question in Chapter 5.

Remark 4.1.2.

- Note that we use the same notation ρ for both a representation of $\widetilde{T}$ and its trivial extension to $\widetilde{B}$.
- One defines the genuine principal series representations of $\widetilde{G'}$ in the same way, by starting with a genuine smooth irreducible representation ρ' of $\widetilde{T'}$.
- We may drop the adjective "genuine" for simplicity. It should be understood that all the representations of the covering groups we consider are genuine, unless otherwise stated.

4.2 Branching Rules for $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$

Let χ be a character of $Z(\widetilde{T})$ and let χ_0 be a fixed extension of χ to A. Let $\rho := \rho_\chi$ be the unique smooth genuine representation of $\widetilde{T}$ with central character χ. The main goal in this section is to decompose $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ into irreducible constituents.

Recall from Lemma 3.3.10 that a typical element of A can be written as $(\mathrm{dg}(a\varpi^{r\underline{n}}),\zeta)$, where $a \in \mathcal{O}^\times$, $r \in \mathbb{Z}$, and a typical element of $\widetilde{T} \cap \widetilde{K}$ can be written as $(\mathrm{dg}(a),\zeta)$, $a \in \mathcal{O}^\times$, $\zeta \in \mu_n$. Define the character

$$\begin{aligned} \vartheta : \mathbb{F}^\times &\to \mu_n \\ a &\mapsto (\varpi, a)_n. \end{aligned} \tag{4.2.1}$$

Observe that by Lemma 2.2.7, ϑ is ramified of degree one. Set $\vartheta_{\mathcal{O}^\times} := \vartheta|_{\mathcal{O}^\times}$.

Lemma 4.2.1. *Let ρ be the unique smooth genuine representation of $\widetilde{T}$ with central character χ. Then*

$$\mathrm{Res}_A \rho \cong \bigoplus_{i=0}^{\underline{n}-1} \chi_i,$$

where the χ_i are $\underline{n}$ distinct characters of A defined by

$$\chi_i\left(\mathrm{dg}(a\varpi^{\underline{n}r}),\zeta\right) = \chi_0\left(\mathrm{dg}(a\varpi^{\underline{n}r}), \vartheta^{2i}_{\mathcal{O}^\times}(a)\zeta\right),$$

for all $a \in \mathcal{O}^\times$, $r \in \mathbb{Z}$, and $0 \le i < \underline{n}$.

Proof: By the Stone-von Neumann theorem, $\rho \cong \mathrm{Ind}_A^{\widetilde{T}}\chi_0$. By Mackey's theorem

$$\mathrm{Res}_A \mathrm{Ind}_A^{\widetilde{T}}\chi_0 = \bigoplus_{s \in S_{\underline{n}}} \mathrm{Ind}^A_{A \cap {}^sA} {\chi_0}^s,$$

where $S_{\underline{n}}$ is a complete set of coset representatives for $A\backslash\widetilde{T}/A$. It is not difficult to see that we can choose

$$S_{\underline{n}} = \{\left(\mathrm{dg}(\varpi^i),1\right) \mid 0 \le i < \underline{n}\}.$$

Since A is stable under conjugation by $S_{\underline{n}}$, $\mathrm{Ind}^A_{A\cap {}^sA}{\chi_0}^s = {\chi_0}^s$. Let $(\mathrm{dg}(a\varpi^{r\underline{n}}),\zeta) \in A$, and $s = (\mathrm{dg}(\varpi^i),1) \in S_{\underline{n}}$. Then

$$\begin{aligned} {\chi_0}^s\left((\mathrm{dg}(a\varpi^{r\underline{n}}),\zeta)\right) &= \chi_0\left(s^{-1}\left(\mathrm{dg}(a\varpi^{r\underline{n}}),\zeta\right)s\right) \\ &= \chi_0\left(\left(\mathrm{dg}(\varpi^{-i}),(\varpi^i,\varpi^i)_n\right)\left(\mathrm{dg}(a\varpi^{r\underline{n}}),\zeta\right)\left(\mathrm{dg}(\varpi^i),1\right)\right) \end{aligned}$$

$$\begin{aligned}
&= \chi_0\left(\left(\mathrm{dg}(a\varpi^{r\underline{n}-i}), (a\varpi^{r\underline{n}}, \varpi^{-i})_n(\varpi^i, \varpi^i)_n\zeta\right)\left(\mathrm{dg}(\varpi^i), 1\right)\right) \\
&= \chi_0\left(\left(\mathrm{dg}(a\varpi^{r\underline{n}}), (\varpi^i, a\varpi^{r\underline{n}-i})_n(a\varpi^{r\underline{n}}, \varpi^{-i})_n(\varpi^i, \varpi^i)_n\zeta\right)\right) \\
&= \chi_0\left(\left(\mathrm{dg}(a\varpi^{r\underline{n}}), (\varpi, a)_n^{2i}\zeta\right)\right) \\
&= \chi_0\left(\left(\mathrm{dg}(a\varpi^{r\underline{n}}), \vartheta^{2i}_{\mathcal{O}^\times}(a)\zeta\right)\right).
\end{aligned}$$

Denote this character by χ_i. To show that the χ_i, $0 \le i < \underline{n}$, are distinct, it is enough to show that $\vartheta^{2i}|_{\mathcal{O}^\times} = 1$ if and only if $i = 0$. By Proposition 2.2.2

$$\vartheta^{2i}(a) = (\varpi, a)_n{}^{2i} = \overline{a^{-1}}^{\frac{(q-1)2i}{n}},$$

which is equal to 1 for all $a \in \mathcal{O}^\times$ if and only if $n|2i$. The result follows. ∎

The characters χ_i defined in Lemma 4.2.1 are clearly distinct when restricted to $\widetilde{T} \cap \widetilde{K}$ and, again writing χ_i for these restrictions,

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\rho = \bigoplus_{i=0}^{\underline{n}-1} \chi_i. \tag{4.2.2}$$

Figure 4.1 gives an inclusion diagram of the subgroups of interest to us. The symbols on the arrows indicate the index of the group on the lower end of the line relative to the one on the upper end.

Proposition 4.2.2. *Let χ_i, $0 \le i < \underline{n}$, denote also the trivial extension of the characters in* (4.2.2) *to $\widetilde{B} \cap \widetilde{K}$. Then*

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho \cong \bigoplus_{i=0}^{\underline{n}-1} \mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i.$$

Proof: By Mackey's theorem we have

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho \cong \bigoplus_{x\in X} \mathrm{Ind}_{x\widetilde{B}x^{-1}\cap\widetilde{K}}^{\widetilde{K}}\mathrm{Res}_{x\widetilde{B}x^{-1}\cap\widetilde{K}}\rho^x,$$

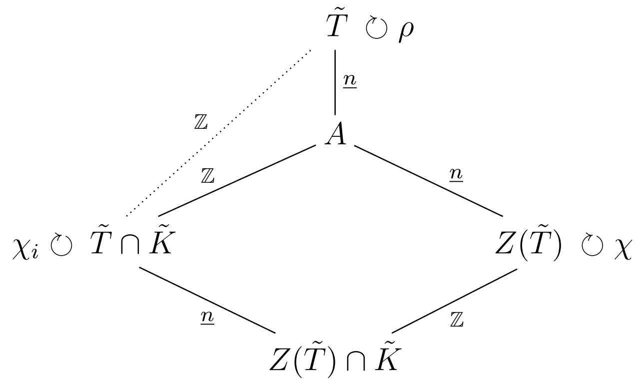


Figure 4.1: Inclusion diagram of the subgroups of $\widetilde{T}$. The symbols on the lines indicate the indices, and the representation of each group is indicated beside the $\circlearrowright$.

where X is a complete set of double coset representatives of $\widetilde{K}$ and $\widetilde{B}$ in $\widetilde{G}$. The Iwasawa decomposition $\widetilde{K}\widetilde{B} = \widetilde{G}$ implies that $X = \{(\mathrm{I}_2, 1)\}$ and hence

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho = \mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\mathrm{Res}_{\widetilde{B}\cap\widetilde{K}}\rho,$$

which by (4.2.2) is equal to

$$\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\left(\bigoplus_{i=0}^{\underline{n}-1}\chi_i\right) = \bigoplus_{i=0}^{\underline{n}-1}\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i.$$

∎

Hence, in order to calculate the K-types, it is enough to decompose each $\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i$, $0 \le i < \underline{n}$, into irreducible representations. Note that the induction space $\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i$ is smooth and admissible: smoothness is a natural consequence of smooth induction and admissibility comes from Lemma 1.1.32. Fix $i \in \{0, \cdots, \underline{n}-1\}$. The smoothness of $\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i$ implies that

$$\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i = \bigcup_{l\ge 1}\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l}.$$

Lemma 4.2.3. *For every $l \geq 1$, $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l}$ is a $\widetilde{K}$-invariant finite-dimensional subspace of $\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i$.*

Proof: By admissibility $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l}$ is finite-dimensional for every $l \geq 1$ and since K_l is normal in $\widetilde{K}$, it is $\widetilde{K}$-invariant. ∎

Hence, to decompose $\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i$ into irreducible constituents, it is enough to decompose each $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l}$ into irreducible constituents.

Definition 4.2.4. *For any character γ of any subgroup D of $\widetilde{T}$, we say γ is* primitive mod m *(or of* depth $m-1$*) if m is the smallest strictly positive integer for which* $\mathrm{Res}_{D\cap K_m}\gamma = 1$.

Note that by Lemma 2.2.11, $Z(\widetilde{T}) \cap K_m = \widetilde{T} \cap K_m$, for all $m \geq 1$. From now on, let $m \geq 1$ be a positive integer such that χ is primitive mod m. Note that since $\chi_i|_{Z(\widetilde{T})} = \chi$, $\chi_i|_{\widetilde{T}\cap K_m} = \chi|_{Z(\widetilde{T})\cap K_m}$. Hence, χ is primitive mod m if and only if the χ_i for $0 \leq i < \underline{n}$ are primitive mod m.

Lemma 4.2.5. *When $0 < l < m$, we have $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l} = \{0\}$.*

Proof: Suppose that f is a vector in this space. Because $\chi_i\big|_{\widetilde{B}\cap K_l} \neq 1$ for $l < m$, we can choose $\mathtt{b} \in \widetilde{B} \cap K_l$ such that $\chi_i(\mathtt{b}) \neq 1$. Let $\mathtt{g} \in \widetilde{K}$. Note that K_l is normal in $\widetilde{K}$ and hence $\mathtt{g}^{-1}\mathtt{b}\mathtt{g} \in K_l$. On the one hand, $f(\mathtt{bg}) = \chi_i(\mathtt{b})f(\mathtt{g})$; on the other hand,

$$f(\mathtt{bg}) = f(\mathtt{g}\mathtt{g}^{-1}\mathtt{bg}) = \left(\mathtt{g}^{-1}\mathtt{bg}\right) \cdot f(\mathtt{g}) = f(\mathtt{g}),$$

since f is fixed by K_l. It follows that $\chi_i(\mathtt{b})f(\mathtt{g}) = f(\mathtt{g})$. Our choice of $\mathtt{b}$ implies that $f(\mathtt{g}) = 0$ and because $\mathtt{g}$ is arbitrary, $f = 0$. ∎

Define

$$\widetilde{B}^l := \widetilde{B} \cap \widetilde{K}/\widetilde{B} \cap K_l \qquad \widetilde{T}^l := \widetilde{T} \cap \widetilde{K}/\widetilde{T} \cap K_l \qquad \widetilde{K}^l := \widetilde{K}/K_l. \tag{4.2.3}$$

Note that the quotient groups $\widetilde{B}^l$, $\widetilde{T}^l$ and $\widetilde{K}^l$ are finite groups. Recall that χ_i is primitive mod m.

Lemma 4.2.6. *The character χ_i factors through $\widetilde{B}^l$ if and only if $l \geq m$.*

Proof: First note that χ_i is determined by its restriction to $\widetilde{T} \cap \widetilde{K}$. By Definition 4.2.4, $\chi_i|_{\widetilde{T}\cap\widetilde{K}}$ is primitive mod m implies that $\chi_i|_{\widetilde{T}\cap K_l} = 1$ for $l \geq m$. Hence, χ_i factors through $\widetilde{T}^l$. Conversely, because χ_i is primitive mod m, by Definition 4.2.4 if χ_i is trivial on $\widetilde{T} \cap K_l$ then $l \geq m$. ∎

We denote the quotient, if exists, by $\bar{\chi}_i$; that is, $\bar{\chi}_i : \widetilde{B}^l \to \mathbb{C}$ is given via

$$\bar{\chi}_i\left(\left(\begin{pmatrix}\bar{t} & \bar{s}\\ 0 & \bar{t}^{-1}\end{pmatrix}, \zeta\right)\right) = \chi_i\left(\left(\begin{pmatrix}t & s\\ 0 & t^{-1}\end{pmatrix}, \zeta\right)\right),$$

where $\bar{t}$ and $\bar{s}$ are images of t and s in $\mathcal{O}^\times/1+\mathfrak{p}^l$ and $\mathcal{O}/\mathfrak{p}^l$ respectively.

Proposition 4.2.7. *Assume $l \geq m$. Then $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l}$ factors to a representation of $\widetilde{K}^l$ isomorphic to $\mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i$.*

Proof: Define the map

$$\begin{aligned}\phi : \left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l} &\to \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i\\ f &\mapsto \phi(f),\end{aligned}$$

where $\phi(f) : \widetilde{K}^l \to \mathbb{C}$ is given by $\mathtt{g}K_l \mapsto f(\mathtt{g})$. Let us show that ϕ is well-defined. Suppose $\mathtt{a} = \mathtt{bg}$ for some $\mathtt{a}, \mathtt{b} \in \widetilde{K}$, and $\mathtt{g} \in K_l$. Then

$$\phi(f)(\mathtt{a}K_l) = f(\mathtt{a}) = f(\mathtt{bg}) = (\mathtt{g}\cdot f)(\mathtt{b}).$$

Since f is fixed by K_l, we have

$$\mathtt{g}\cdot f(\mathtt{b}) = f(\mathtt{b}) = \phi(f)(\mathtt{b}K_l),$$

so ϕ is well-defined. Note that ϕ is clearly injective. To see it is surjective suppose $\bar{f} \in \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i$. Define a function $f : \widetilde{K} \to \mathbb{C}$ via $\mathtt{g} \mapsto \bar{f}(\mathtt{g}K_l)$. Observe that for every $\mathtt{b} \in \widetilde{B}$ and $\mathtt{g} \in \widetilde{K}$,

$$f(\mathtt{bg}) = \bar{f}(\mathtt{bg}K_l) = \bar{\chi}_i(\mathtt{b}K_l)\bar{f}(\mathtt{g}K_l) = \chi_i(\mathtt{b})f(\mathtt{g}),$$

and for $\mathtt{k} \in K_l$ and $\mathtt{g} \in \widetilde{K}$ we have,

$$\mathtt{k} \cdot f(\mathtt{g}) = f(\mathtt{gk}) = \bar{f}(\mathtt{gk}K_l) = f(\mathtt{g}).$$

Hence $f \in \left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_l}$ and $\phi(f) = \bar{f}$. Moreover, to see that the map ϕ commutes with the action of $\widetilde{K}$ and $\widetilde{K}^l$, observe that for $\mathtt{g} \in \widetilde{K}$ and $\mathtt{a}K_l \in \widetilde{K}^l$,

$$\phi(\mathtt{g} \cdot f)(\mathtt{a}K_l) = \mathtt{g} \cdot f(\mathtt{a}) = f(\mathtt{ag}) = \phi(f)(\mathtt{ag}K_l) = (\mathtt{g}K_l) \cdot \phi(f)(\mathtt{a}K_l).$$

∎

Proposition 4.2.7 tells us that, in order to decompose $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_l}$ into irreducible constituents, it is enough to decompose $\mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i$. The latter, being a representation of the finite group $\widetilde{K}^l$, is completely reducible. Hence, we are interested in counting the dimension of $\mathrm{Hom}(\mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i, \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i)$. Let $\mathcal{H} := \mathcal{H}(\widetilde{B}^l\backslash\widetilde{K}^l/\widetilde{B}^l, \bar{\chi}_i, \bar{\chi}_i)$ be the Hecke algebra defined in Definition 1.1.36. By Proposition 1.1.38, it is readily inferred that

$$\dim \mathrm{Hom}(\mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i, \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\bar{\chi}_i) = \dim \mathcal{H}(\widetilde{B}^l\backslash\widetilde{K}^l/\widetilde{B}^l, \bar{\chi}_i, \bar{\chi}_i). \tag{4.2.4}$$

Hence, it suffices to count the dimension of $\mathcal{H}$. A basis of $\mathcal{H}$ consists of functions transforming on the left and right by $\bar{\chi}_i$, with support on a double coset of $\widetilde{B}^l$ in $\widetilde{K}^l$.

Lemma 4.2.8. *Suppose $\{\mathbf{g}_\alpha\}_{\alpha\in\Omega}$ is a complete set of double coset representatives of B^l in K^l. Then $\{(\mathbf{g}_\alpha, 1)\}$ is a complete set of representatives for the double cosets of $\widetilde{B}^l$ in $\widetilde{K}^l$.*

Proof: Note that if $(\mathbf{b}, \zeta)(\mathbf{g}_\gamma, 1)(\mathbf{b}', \zeta') = (\mathbf{g}_\beta, 1)$ for $\mathbf{b}, \mathbf{b}' \in B^l$, $\zeta, \zeta' \in \mu_n$ and $\gamma, \beta \in \Omega$, then $\mathbf{b}\mathbf{g}_\gamma\mathbf{b}' = \mathbf{g}_\beta$. Hence, $\{(\mathbf{g}_\alpha, 1)\}_{\alpha\in\Omega}$ represents distinct cosets. For an arbitrary $(\mathbf{k}, \zeta) \in \widetilde{K}^l$, there exists $\mathbf{b}, \mathbf{b}' \in K^l$ such that $\mathbf{bkb}' = \mathbf{g}_\alpha$ for some double coset representative $\mathbf{g}_\alpha$. Set $\mathtt{b} = (\mathbf{b}, \beta(\mathbf{b}, \mathbf{k})^{-1}\zeta^{-1})$ and $\mathtt{b}' = (\mathbf{b}', \beta(\mathbf{bk}, \mathbf{b}')^{-1})$. Then

$$\mathtt{b}(\mathbf{k}, \zeta)\mathtt{b}' = (\mathbf{g}_\alpha, 1).$$

So, the set $\{(\mathbf{g}_\alpha, 1)\}_{\alpha\in\Omega}$ of representatives is complete. ∎

Recall that $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathrm{lt}(m) = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$ for all $m \in \mathbb{F}$. The complete set of representatives for double cosets of B^l in K^l

$$\{\mathrm{I}_2, w, \mathrm{lt}(x\varpi^r) \mid x \in \{1, \varepsilon\},\ 1 \leq r < l\}, \tag{4.2.5}$$

where ε is a fixed non-square in $\mathcal{O}^\times/\mathcal{O}^{\times 2}$, is given in [Nev05]. Set $\widetilde{w} := (w, 1)$ and $\widetilde{\mathrm{lt}}(m) := (\mathrm{lt}(m), 1)$ for all $m \in \mathbb{F}$. Lemma 4.2.8 implies that

$$\{(\mathrm{I}_2, 1), \widetilde{w}, \widetilde{\mathrm{lt}}(x\varpi^r) \mid x \in \{1, \varepsilon\}, 1 \leq r < l\} \tag{4.2.6}$$

is a complete set of double coset representatives for $\widetilde{B}^l$ in $\widetilde{K}^l$. Recall that $(T\cap K)\times\{1\}$ is a subgroup of $\widetilde{T}$, which we identify with $T \cap K$. Set $(T\cap K)^2 := \{\mathrm{dg}(t^2) \mid t \in \mathcal{O}^\times\}$. Moreover, set $T^l := \{\iota(t) \mid t \in \mathcal{O}^\times/1+\mathfrak{p}^l\}$ and $(T^l)^2 := \{\iota(t^2) \mid t \in \mathcal{O}^\times/1+\mathfrak{p}^l\}$. It is not difficult to see that T^l and $(T^l)^2$ are subgroups of $\widetilde{T}^l$.

Proposition 4.2.9. *The dimension of $\mathcal{H}$, $l \geq m$, is*

$$\dim\mathcal{H} = \begin{cases} 1 + 2(l-m), & \textit{if} \quad \chi_i|_{(T\cap K)^2} \neq 1; \\ 2l, & \textit{otherwise.} \end{cases}$$

Proof: Assume $l \geq m$. Recall from Definition 1.1.36 that $f(\mathtt{bkb}') = \bar{\chi}_i(\mathtt{b})f(\mathtt{k})\bar{\chi}_i(\mathtt{b}')$ for all $f \in \mathcal{H}$, $\mathtt{b}, \mathtt{b}' \in \widetilde{B}^l$ and $\mathtt{k} \in \widetilde{K}^l$. Hence, for every double coset representative $\mathtt{x}$ in (4.2.6), there exists a function $f \in \mathcal{H}$, with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{bxb}' = \mathtt{x}$ implies that $\bar{\chi}_i(\mathtt{bb}') = 1$ for all $\mathtt{b}, \mathtt{b}' \in \widetilde{B}^l$. The set of such double cosets parameterizes a basis for $\mathcal{H}$. We now determine these double cosets. Let

$$\mathtt{b} = (\mathbf{b}, \zeta) = \left(\begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix}, \zeta\right) \quad \text{and} \quad \mathtt{b}' = (\mathbf{b}', \zeta') = \left(\begin{pmatrix} t' & s' \\ 0 & t'^{-1} \end{pmatrix}, \zeta'\right),$$

where $t, t' \in \mathcal{O}^\times/1+\mathfrak{p}^l$, $s, s' \in \mathcal{O}/\mathfrak{p}^l$ and $\zeta, \zeta' \in \mu_n$ denote arbitrary elements of $\widetilde{B}^l$.

The identity coset $\widetilde{B}^l$: A function $f \in \mathcal{H}$ has support on $\widetilde{B}^l$ if and only if $f(\mathtt{b}) = \bar{\chi}_i(\mathtt{b})$, $\forall \mathtt{b} \in \widetilde{B}^l$. So there is always a function with support on the identity coset, namely $f = \bar{\chi}_i$.

The coset of $\widetilde{w}$: For $\mathtt{b}$ and $\mathtt{b}'$ in $\widetilde{B}^l$,

$$\mathtt{b}\mathtt{w}\mathtt{b}' = \mathtt{w} \tag{4.2.7}$$

implies, via a quick calculation, that $\mathbf{b} = \mathbf{b}' = \mathrm{dg}(t)$, for some $t \in \mathcal{O}^\times/1+\mathfrak{p}^l$ and $\zeta' = \zeta^{-1}$. Therefore,

$$\bar{\chi}_i(\mathtt{b}\mathtt{b}') = \bar{\chi}_i\left((\mathrm{dg}(t),\zeta)(\mathrm{dg}(t),\zeta^{-1})\right) = \bar{\chi}_i\left(\mathrm{dg}(t^2),(t,t)_n\right) = \bar{\chi}_i\left(\mathrm{dg}(t^2),1\right),$$

and so, $\mathcal{H}$ contains a function with support on this coset if and only if $\bar{\chi}_i(\iota(t^2)) = 1$ for all $t \in \mathcal{O}^\times/1+\mathfrak{p}^l$, that is, if and only if $\bar{\chi}_i\big|_{(T^l)^2} = 1$. Finally, observe that $\bar{\chi}_i\big|_{(T^l)^2} = 1$, $l \geq m$, and $0 \leq i < \underline{n}$ if and only if $\chi_i|_{(T\cap K)^2} = 1$. Suppose $\chi_i|_{(T\cap K)^2} = 1$, for some $0 \leq i < \underline{n}$. We show that in this case, $m = 1$. Suppose $\alpha \in 1+\mathfrak{p}$; consider $f(X) = X^2 - \alpha$. Observe that $f(1) = 0 \mod \mathfrak{p}$, and $f'(1) = 2(1) \neq 0 \mod p$. By Hensel's lemma, $f(X)$ has a root in $\mathcal{O}$; that is $\alpha \in \mathcal{O}^{\times 2}$. Therefore $1+\mathfrak{p} \subset \mathcal{O}^{\times 2}$, which implies $\chi_i|_{\widetilde{T}\cap K_1} = 1$, so $m = 1$.

The coset of $\widetilde{\mathrm{lt}}(x\varpi^r)$: For $\mathtt{b}$ and $\mathtt{b}'$ in $\widetilde{B}^l$

$$\mathtt{b}\,\widetilde{\mathrm{lt}}(x\varpi^r)\mathtt{b}' = \widetilde{\mathrm{lt}}(x\varpi^r) \tag{4.2.8}$$

implies that $tt' \in 1+\mathfrak{p}^r$ and $\zeta = \zeta'^{-1}$. Therefore,

$$\bar{\chi}_i(\mathtt{b}\mathtt{b}') = \bar{\chi}_i(\mathbf{b}\mathbf{b}',1) = \bar{\chi}_i(\begin{pmatrix} tt' & ts'+st'^{-1} \\ 0 & t^{-1}t'^{-1} \end{pmatrix},1).$$

Note that $\begin{pmatrix} tt' & ts'+st'^{-1} \\ 0 & t^{-1}t'^{-1} \end{pmatrix} \in \widetilde{B}\cap K_r$. Hence, $\bar{\chi}_i(\mathtt{b}\mathtt{b}') = 1$ if and only if $\widetilde{B}\cap K_r \subseteq \ker(\chi_i)$. The latter holds if and only if $r \geq m$, since χ_i is primitive mod m.

Now, let us summarize our result. There is always one function with support on the identity coset, and $2(l-m)$ functions on cosets represented by $\widetilde{\mathrm{lt}}(x\varpi^r)$, $x \in \{1, \varepsilon\}$, $m \leq r < l$. If $\chi_i\big|_{(T\cap K)^2} \neq 1$, no function in $\mathcal{H}$ has support on the double coset represented by $\widetilde{w}$, otherwise, there exists an additional function in $\mathcal{H}$ with support on the double coset represented by $\widetilde{w}$. ∎

We elaborate on the condition $\chi_i|_{(T\cap K)^2} = 1$ that appears in Proposition 4.2.9.

Lemma 4.2.10. *For each $0 \leq i < \underline{n}$, $\chi_i|_{(T\cap K)^2} = 1$ if and only if $\chi_0|_{(T\cap K)^2} = \epsilon\circ\vartheta|^{-2i}_{\mathcal{O}^{\times 2}}$.*

Proof: Let $\iota(s) \in (T\cap K)^2$, so $s \in \mathcal{O}^{\times 2}$. By Lemma 4.2.1

$$\chi_i(\iota(s)) = \chi_0\left(\mathrm{dg}(s), \vartheta(s)^{2i}\right) = \chi_0(\iota(s))\epsilon(\vartheta(s)^{2i}),$$

which is equal to 1 if and only if $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta|^{-2i}_{\mathcal{O}^{\times 2}}$. ∎

Lemma 4.2.11. *If $4 \nmid n$ then the characters $\vartheta|^{-2i}_{\mathcal{O}^{\times 2}}$, $0 \leq i < \underline{n}$ are distinct. Otherwise, the $\vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, $0 \leq i < \frac{n}{4}$, are distinct; for $\frac{n}{4} \leq j < \frac{n}{2}$, $\vartheta^{-2j}_{\mathcal{O}^{\times 2}} = \vartheta^{-2(j-\frac{n}{4})}_{\mathcal{O}^{\times 2}}$.*

Proof: By definition of ϑ in (4.2.1), $\vartheta^{-2i}(s) = 1$ for all $s \in \mathcal{O}^{\times 2}$ if and only if $\overline{t^2}^{\frac{(q-1)2i}{n}} = 1$ for all $t \in \mathcal{O}^{\times}$, or equivalently when $n|4i$. Therefore, $i = 0$ unless $4|n$, in which case the equality holds for both $i = 0$ and $i = n/4$. ∎

We summarize the result in the following corollary.

Corollary 4.2.12. *For $l > m$, the l-level representations*

$$\widetilde{W}_{i,l} := (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_l}/(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_{l-1}}$$

decompose into two inequivalent irreducible subrepresentations. The m-level representations $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_m}$, $0 \leq i < \underline{n}$, are irreducible except when $m = 1$ and there exists $0 \leq k < \underline{n}$ such that $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta^{-2k}_{\mathcal{O}^{\times 2}}$; in this case, we are in one of the following situations:

1. *If $4 \nmid n$ then $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_1}$ is irreducible for all $0 \le i < \underline{n}$ such that $i \neq k$, and for $i = k$, it decomposes into two inequivalent irreducible constituents.*

2. *If $4|n$ then $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_1}$ is irreducible for all $0 \le i < \underline{n}$ such that $i \neq k$ and $i \neq k + \frac{n}{4} \mod \underline{n}$, and for $i = k$ or $i = k + \frac{n}{4} \mod \underline{n}$, it decomposes into two inequivalent irreducible constituents.*

Proof: It follows from Proposition 4.2.7, (4.2.4) and Proposition 4.2.9, that for $l > m$

$$\dim \mathrm{Hom}(\widetilde{W}_{i,l}, \widetilde{W}_{i,l}) = 2.$$

Hence, $\widetilde{W}_{i,l}$ decomposes into two inequivalent irreducible subrepresentations. Moreover,

$$\dim \mathrm{Hom}(\left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{K}}\chi_i\right)^{K_m}, \left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{K}}\chi_i\right)^{K_m}) = \begin{cases} 1, & \text{if } \chi_i\big|_{(T\cap K)^2} \neq 1 \\ 2, & \text{otherwise.} \end{cases} \tag{4.2.9}$$

By Lemma 4.2.10, $\chi_i\big|_{(T\cap K)^2} = 1$ is equivalent to $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-2i}$, which also implies that $m = 1$. Hence, $\left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{K}}\chi_i\right)^{K_m}$ is irreducible except when $m = 1$ and $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-2i}$, where it decomposes into two irreducible constituents. If the latter is the case, by Lemma 4.2.11, there is exactly one $0 \le i < \underline{n}$ satisfying $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-2i}$ if $4 \nmid n$, and there are exactly two $0 \le i < \underline{n}$ satisfying $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-2i}$ if $4|n$. ∎

For each $0 \le i < \underline{n}$, let $\widetilde{W}_{i,l}^+$ and $\widetilde{W}_{i,l}^-$ denote two irreducible inequivalent spaces such that $\widetilde{W}_{i,l} \cong \widetilde{W}_{i,l}^+ \oplus \widetilde{W}_{i,l}^-$, as in Corollary 4.2.12. This corollary can be rephrased directly as follows:

Corollary 4.2.13. *Assume $l \geq m$. We can decompose $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ as follows:*

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho \cong \bigoplus_{i=0}^{\underline{n}-1}\left((\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_m} \oplus \bigoplus_{l>m}\left(\widetilde{W}_{i,l}^+ \oplus \widetilde{W}_{i,l}^-\right)\right).$$

All the pieces are irreducible except when $m = 1$ and $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$ for some $0 \leq i < \underline{n}$ then

1. *If $4 \nmid n$ then there is exactly one $0 \leq i < \underline{n}$ for which $(\mathrm{Ind}^{\widetilde{K}}_{\widetilde{B\cap K}}\chi_i)^{K_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

2. *If $4|n$ then there are exactly two $0 \leq i, k < \underline{n}$, $|i-k| = \frac{n}{4}$ for which $(\mathrm{Ind}^{\widetilde{K}}_{\widetilde{B\cap K}}\chi_i)^{K_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

Next we determine the multiplicity of each constituent in the decomposition in Corollary 4.2.13. To do so, we count the dimension of

$$\mathrm{Hom}_{\widetilde{K}^l}\left(\mathrm{Ind}^{\widetilde{K}^l}_{\widetilde{B}^l}\bar{\chi}_k, \mathrm{Ind}^{\widetilde{K}^l}_{\widetilde{B}^l}\bar{\chi}_i\right).$$

By Proposition 4.2.7 and Proposition 1.1.38, it is enough to count the dimension of the Hecke algebra $\mathcal{H}(\widetilde{B}^l\backslash\widetilde{K}^l/\widetilde{B}^l, \bar{\chi}_k, \bar{\chi}_i)$.

Proposition 4.2.14. *Let $l \geq m$ and $\mathcal{H}_{k,i} = \mathcal{H}(\widetilde{B}^l\backslash\widetilde{K}^l/\widetilde{B}^l, \bar{\chi}_k, \bar{\chi}_i)$, where $0 \leq i, k < \underline{n}$. Then $\dim \mathcal{H}_{k,i}$ is*

$$\begin{cases} 2l-1, & \text{if } \chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta^{-(k+i)}_{\mathcal{O}^{\times 2}} \\ 2(l-m), & \text{otherwise.} \end{cases}$$

Proof: Similar to the proof of Proposition 4.2.9, we determine which double cosets in $\widetilde{B}^l\backslash\widetilde{K}^l/\widetilde{B}^l$ support a function in $\mathcal{H}_{k,i}$. For every double coset representative $\mathtt{x}$ in Lemma 4.2.8, there exists a function $f \in \mathcal{H}_{k,i}$ with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{b}\mathtt{x}\mathtt{b}' = \mathtt{x}$, $\mathtt{b}, \mathtt{b}' \in \widetilde{B}^l$, implies that $\bar{\chi}_k(\mathtt{b})\bar{\chi}_i(\mathtt{b}') = 1$. Let $\mathtt{b} = (\mathbf{b}, \zeta) = \left(\left(\begin{smallmatrix} t & s \\ 0 & t^{-1}\end{smallmatrix}\right), \zeta\right)$ and $\mathtt{b}' = (\mathbf{b}', \zeta') = \left(\left(\begin{smallmatrix} t' & s' \\ 0 & t'^{-1}\end{smallmatrix}\right), \zeta'\right)$ be arbitrary elements of $\widetilde{B}^l$.

Because $\chi_k \neq \chi_i$, there is no function in $\mathcal{H}_{k,i}$ with support on the identity double coset.

For the double coset of $\widetilde{w}$, $\mathtt{b}\widetilde{w}\mathtt{b}' = \widetilde{w}$ implies that $\mathbf{b} = \mathbf{b}' = \mathrm{dg}(t)$, for some $t \in \mathcal{O}^\times/1+\mathfrak{p}^l$ and $\zeta' = \zeta^{-1}$. Therefore, $\bar{\chi}_k(\mathtt{b})\bar{\chi}_i(\mathtt{b}') = \bar{\chi}_k(\mathrm{dg}(t), \zeta)\,\bar{\chi}_i(\mathrm{dg}(t), \zeta^{-1})$

equals

$$\begin{aligned}\bar{\chi}_0\left(\mathrm{dg}(t),\vartheta(t)^{2k}\zeta\right)\bar{\chi}_0\left(\mathrm{dg}(t),\vartheta(t)^{2i}\zeta^{-1}\right) &= \bar{\chi}_0\left(\mathrm{dg}(t^2),\vartheta(t)^{2k+2i}\right)\\ &= \bar{\chi}_0\left(\iota(t^2)(\mathrm{I}_2,\vartheta(t^2)^{k+i})\right)\\ &= \bar{\chi}_0\left(\iota(t^2)\right)\epsilon\left(\vartheta(t^2)^{k+i}\right).\end{aligned}$$

Therefore, $\bar{\chi}_k(\mathsf{b})\bar{\chi}_i(\mathsf{b}')=1$ if and only if $\chi_0|_{(T\cap K)^2}=\epsilon\circ\vartheta_{\mathcal{O}^{\times 2}}^{-(k+i)}$. In this case, $m=1$ and $\widetilde{w}$ supports a function in $\mathcal{H}_{k,i}$.

Finally, for the double cosets represented by $\widetilde{\mathrm{lt}}(x\varpi^r)$, $x\in\{1,\varepsilon\}$, $1\le r<l$, $\mathsf{b}\,\widetilde{\mathrm{lt}}(x\varpi^r)\mathsf{b}'=\widetilde{\mathrm{lt}}(x\varpi^r)$ implies that $\zeta'=\zeta^{-1}$, and $t+s\varpi^r=t'^{-1}\mod\mathfrak{p}^l$, or equivalently, $t=t'^{-1}\mod\mathfrak{p}^r$, and $t^{-1}\varpi^r=\varpi^r t'^{-1}\mod\mathfrak{p}^l$, or equivalently $t^{-1}=t'^{-1}\mod\mathfrak{p}^{l-r}$. Observe that, in general, $\bar{\chi}_k(\mathsf{b})\bar{\chi}_i(\mathsf{b}')$ is equal to

$$\begin{aligned}\bar{\chi}_k\left(\mathrm{dg}(t),\zeta\right)\bar{\chi}_i\left(\mathrm{dg}(t'),\zeta'\right) &= \chi_0\left(\mathrm{dg}(t),\vartheta(t)^{2k}\zeta\right)\chi_0\left(\mathrm{dg}(t'),\vartheta(t')^{2i}\zeta'\right) \quad (4.2.10)\\ &= \chi_0\left(\mathrm{dg}(tt'),\vartheta(t)^{2k}\vartheta(t')^{2i}\zeta\zeta'\right)\\ &= \chi_0\left(\iota(tt')\right)\epsilon\left(\vartheta(t)^{2k}\vartheta(t')^{2i}\zeta\zeta'\right).\end{aligned}$$

Lemma 2.2.7 implies that ϑ is primitive mod one. Observe that $r\ge 1$ and $l-r\ge 1$. Therefore, $t=t'^{-1}\mod\mathfrak{p}$ and $t=t'\mod\mathfrak{p}$, which implies that $t=t'=\alpha\mod\mathfrak{p}$ where $\alpha\in\{\pm 1\}$. Hence, $\vartheta(t)^2=\vartheta(t')^2=1$, and (4.2.10) simplifies to $\bar{\chi}_0\left(\iota(tt')\right)\epsilon(\zeta\zeta')$. We are in one of the following situations:

Case 1: Suppose $r\ge m$. Then we have $\zeta'=\zeta^{-1}$, and $t=t'^{-1}\mod\mathfrak{p}^m$; that is $tt'\in 1+\mathfrak{p}^m$. Hence, $\bar{\chi}_0\left(\iota(tt')\right)\epsilon(\zeta\zeta')=\bar{\chi}_0(tt')=1$, because χ_0 is primitive mod m. Therefore, in this case, there is always a function in $\mathcal{H}_{k,i}$ with support on these double cosets.

Case 2: Suppose $r<m$. Then $\zeta'=\zeta^{-1}$, so $\bar{\chi}_0\left(\iota(tt')\right)\epsilon(\zeta\zeta')=\bar{\chi}_0(tt')$, which equals one if and only if $tt'\in 1+\mathfrak{p}^m$, which is not the case in general. Hence, in this case, there is no function in $\mathcal{H}_{k,i}$ with support on these double cosets.

To summarize the result, the coset represented by $\widetilde{w}$ supports a function in $\mathcal{H}_{k,i}$ if and only if $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta_{\mathcal{O}^{\times 2}}^{-(k+i)}$. If $r \geq m$ then the cosets represented by $\mathrm{lt}(x\varpi^r)$ support a function in $\mathcal{H}_{k,i}$; otherwise, there is no function in $\mathcal{H}_{k,i}$ with support on these double cosets. ∎

Corollary 4.2.15. *In the decomposition of* $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ *given in Corollary 4.2.13,*

1. *For each* $0 \leq i < \underline{n}$ *and* $l > m$, *there exists a way of decomposing* $\widetilde{W}_{i,l}$ *as* $\widetilde{W}_{i,l}^{+} \oplus \widetilde{W}_{i,l}^{-}$ *such that for* $l > m$, $\widetilde{W}_{i,l}^{+} \cong \widetilde{W}_{j,l}^{+}$ *and* $\widetilde{W}_{i,l}^{-} \cong \widetilde{W}_{j,l}^{-}$ *for all* $0 \leq i, j < \underline{n}$.
2. *For* $m = l$, $\{(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_m} \mid 0 \leq i < \underline{n}\}$ *consists of mutually inequivalent representations, except when* $m = 1$ *and* $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta|_{\mathcal{O}^{\times 2}}^{-j}$, *for some* $0 \leq j < \underline{n}$, *where*
$$\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{K_1} \cong \left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_k\right)^{K_1},$$
exactly when $i + k \equiv j \mod \underline{n}$.

Proof: It follows from Proposition 4.2.14 that for $l > m$,
$$\dim \mathrm{Hom}_{\widetilde{K}}\left(\widetilde{W}_{i,l}, \widetilde{W}_{k,l}\right) = 2$$
and when $i + k \equiv j \mod \underline{n}$
$$\dim \mathrm{Hom}_{\widetilde{K}}\left((\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_m}, (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_k)^{K_m}\right) = \begin{cases} 1, & \chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta|_{\mathcal{O}^{\times 2}}^{-j} \\ 0, & \text{otherwise,} \end{cases}$$
and hence the result. ∎

Example 4.2.16. *If* $\chi_0|_{(T\cap K)^2}$ *is trivial, Corollary 4.2.15 implies that* $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{K_1}$ *appears with multiplicity one in the decomposition of* $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$, *and by Corollary 4.2.13 decomposes into two irreducible constituents.*

In order to further investigate the irreducible spaces $\widetilde{W}_{i,l}^{+}$ and $\widetilde{W}_{i,l}^{-}$, we will show that $\widetilde{W}_{i,l}$, $0 \leq i < \underline{n}$ is the restriction to $\widetilde{K}$ of an irreducible representation of $\widetilde{K'}$, whence it follows that $\widetilde{W}_{i,l}^{+}$ and $\widetilde{W}_{i,l}^{-}$ have the same degree. This motivates us to study the genuine principal series representations of $\widetilde{G'}$ and their branching rules.

4.3 Branching Rules for $\mathrm{Res}_{\widetilde{K'}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$

We now apply the same machinery to $\widetilde{G'}$. Let χ' be a genuine character of $Z(\widetilde{T'})$ and let χ_0' be a fixed extension of χ' to A'. Let $(\rho', \mathrm{Ind}_{A'}^{\widetilde{T'}}\chi_0')$ be the unique irreducible representation of $\widetilde{T'}$ with central character χ', trivially extended to a representation of $\widetilde{B'}$. We will decompose $\mathrm{Res}_{\widetilde{K'}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$ into irreducible constituents. The arguments in this section are partly analogous to those in Section 4.2. To avoid repetition, we will occasionally refer the reader to arguments in Section 4.2.

Recall that $A' = C_{\widetilde{T'}}(\widetilde{T} \cap \widetilde{K'})$ is a maximal abelian subgroup of $\widetilde{T'}$. Recall from Proposition 3.3.14 that a typical element of A' can be written as $(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta)$, where $a, b \in \mathcal{O}^\times$, $u, v \in \mathbb{Z}$, and $\zeta \in \mu_n$, and a typical element of $\widetilde{T'} \cap \widetilde{K'}$ can be written as $(\mathrm{dg}(a, b), \zeta)$, for some $a, b \in \mathcal{O}^\times$, $\zeta \in \mu_n$.

Lemma 4.3.1. *Let ρ' be the unique smooth genuine representation of $\widetilde{T'}$ with central character χ'. Then*

$$\mathrm{Res}_{A'}\rho' \cong \bigoplus_{i,j=0}^{n-1} \chi_{i,j}',$$

where the $\chi_{i,j}'$ denote n^2 distinct characters of A', defined by

$$\chi_{i,j}'\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta\right) = \chi_0'\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \vartheta(a)^{-j}\vartheta(b)^{-i}\zeta\right)$$

where $a, b \in \mathcal{O}^\times$, $u, v \in \mathbb{Z}$ and $\zeta \in \mu_n$ and $\vartheta(a) = (\varpi, a)_n$ was defined in (4.2.1).

Proof: Observe that $\mathcal{S} = \{(\mathrm{dg}(\varpi^i, \varpi^j), 1) \mid 0 \leq i, j < n\}$ is a complete set of coset representatives for $A' \backslash \widetilde{T'} / A'$. By Mackey's theorem

$$\mathrm{Res}_{A'}\mathrm{Ind}_{A'}^{\widetilde{T'}}\chi_0' = \bigoplus_{\ddot{s} \in \mathcal{S}} \mathrm{Ind}_{A' \cap {}^{\ddot{s}}A'}^{A'}{\chi_0'}^{\ddot{s}}.$$

One can see that A' is stable under conjugation by elements of $\mathcal{S}$, so $\mathrm{Ind}_{A'\cap^{\ddot{s}}A'}^{A'}{\chi'_0}^{\ddot{s}} = {\chi'_0}^{\ddot{s}}$. Let $(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta) \in A'$ where $a, b \in \mathcal{O}^\times$ and $\zeta \in \mu_n$, and let $\ddot{s} = (\mathrm{dg}(\varpi^i, \varpi^j), 1) \in \mathcal{S}$. Note that

$$\begin{aligned}
\ddot{s}^{-1}\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta\right)\ddot{s} &= \left(\mathrm{dg}(\varpi^{-i}, \varpi^{-j}), (\varpi^i, \varpi^j)_n\right)\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta\right)\left(\mathrm{dg}(\varpi^i, \varpi^j), 1\right) \\
&= \left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta(b, \varpi)_n^i (a, \varpi)_n^j\right) \\
&= \left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \vartheta(a)^{-j}\vartheta(b)^{-i}\zeta\right).
\end{aligned}$$

Therefore,

$${\chi'_0}^{\ddot{s}}\left(\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \zeta\right)\right) = \chi'_0\left(\mathrm{dg}(a\varpi^{un}, b\varpi^{vn}), \vartheta(a)^{-j}\vartheta(b)^{-i}\zeta\right).$$

Denote these characters by $\chi'_{i,j}$. A similar argument to the one in Lemma 4.2.1 implies that $\{\chi'_{i,j} \mid 0 \leq i, j < n\}$ is a set of n^2 distinct characters of A'. ∎

It follows from Lemma 4.3.1 that the characters $\chi'_{i,j}|_{\widetilde{T'}\cap\widetilde{K'}}$ are also distinct, and writing again $\chi'_{i,j}$ for these restrictions,

$$\mathrm{Res}_{\widetilde{T'}\cap\widetilde{K'}}\rho' \cong \bigoplus_{i,j=0}^{n-1} \chi'_{i,j}. \tag{4.3.1}$$

Proposition 4.3.2. *Let ρ' be the unique smooth genuine representation of $\widetilde{T'}$ with central character χ', trivially extended to $\widetilde{B'}$. Then*

$$\mathrm{Res}_{\widetilde{K'}}(\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho') \cong \bigoplus_{i,j=0}^{n-1} \mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j}.$$

The proof is similar to that of Proposition 4.2.2. It follows from Lemma 1.1.32 that, for every $0 \leq i, j < n$, $\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j}$ is smooth and admissible. Fix a pair (i, j), $0 \leq i, j < n$. We have,

$$\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j} = \bigcup_{l\geq 1}\left(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j}\right)^{K'_l}.$$

Definition 4.3.3. *For any character γ of any subgroup D of $\widetilde{T'}$, we say γ is primitive mod m (or of depth $m-1$) if m is the smallest strictly positive integer for which* $\mathrm{Res}_{D\cap K'_m}\gamma = 1$.

Suppose χ' is primitive mod m. It follows that the $\chi'_{i,j}$ are also primitive mod m. One can define the finite quotient groups $\widetilde{B'}^l$ and $\widetilde{K'}^l$ analogously to (4.2.3), and as in Lemma 4.2.6, $\chi'_{i,j}$ factors through $\widetilde{B'}^l$ if and only if $l \geq m$. We will denote the quotient, if exists, by $\bar{\chi}'_{i,j}$.

Proposition 4.3.4. *Let $\chi'_{i,j}$ be primitive mod m. If $l < m$, $\left(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j}\right)^{K'_l} = \{0\}$. If $l \geq m$, $\left(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j}\right)^{K'_l}$ factors to a representation of $\widetilde{K'}^l$ isomorphic to $\mathrm{Ind}_{\widetilde{B'}^l}^{\widetilde{K'}^l}\bar{\chi}'_{i,j}$.*

The proof is similar to that of Proposition 4.2.7 and Lemma 4.2.5.

Fix $l \geq m$. By Proposition 4.3.4 and Proposition 1.1.38, our next goal is to compute the dimension of the algebra $\mathcal{H}' := \mathcal{H}'(\widetilde{B'}^l\backslash\widetilde{K'}^l/\widetilde{B'}^l, \bar{\chi}'_{i,j}, \bar{\chi}'_{i,j})$. A basis of $\mathcal{H}'$ consists of functions with support on double cosets of $\widetilde{B'}^l$ in $\widetilde{K'}^l$ that transform on both sides by $\bar{\chi}'_{i,j}$.

Lemma 4.3.5. *A complete set of double coset representatives of $\widetilde{B'}^l$ in $\widetilde{K'}^l$ is given by*

$$\{(\mathrm{I}_2, 1), \widetilde{w}, \widetilde{\mathrm{lt}}(\varpi^r) \mid 1 \leq r < l\}.$$

Proof: Note that this set is a subset of the set $\mathcal{A}$ of double coset representatives of $\widetilde{B}^l\backslash\widetilde{K}^l/\widetilde{B}^l$ in (4.2.6). Observe that under the isomorphism

$$\begin{aligned} \mathbb{F}^\times \rtimes \widetilde{G} &\cong \widetilde{G'} \\ (y, (\mathbf{g}, \zeta)) &\mapsto (\mathrm{dg}(1, y)\mathbf{g}, \zeta), \end{aligned} \tag{4.3.2}$$

$\mathcal{O}^\times \times \widetilde{K^l}$ maps to $\widetilde{K'}^l$ and $\mathcal{O}^\times \times \widetilde{B}^l$ maps to $\widetilde{B}'^l$. For every $\mathtt{k}' \in \widetilde{K'}^l$, let $(y, \mathtt{k})$ be the inverse image of $\mathtt{k}'$ under the isomorphism (4.3.2), and let $\mathtt{b}_1, \mathtt{b}_2 \in \widetilde{B}^l$ be such that $\mathtt{b}_1\mathtt{x}\mathtt{b}_2 = \mathtt{k}$, for some $\mathtt{x} \in \mathcal{A}$. Let $\mathtt{b}'_1$ and $\mathtt{b}'_2$ be the image of $(y, \mathtt{b}_1)$ and $(y, \mathtt{b}_2)$ under (4.3.2) respectively. It follows from the multiplication of $\mathbb{F}^\times \rtimes \widetilde{G}$ given in (3.1.3), and the isomorphism map (4.3.2), that $\mathtt{b}_1'\mathtt{x}\mathtt{b}_2' = \mathtt{k}'$. Thus, $\widetilde{K'}^l = \bigcup_{\mathtt{x}\in\mathcal{A}} \widetilde{B'}^l\mathtt{x}\widetilde{B'}^l$. A short calculation shows that

$$\left(\mathrm{dg}(\varepsilon^{-1},1),1\right)\widetilde{\mathrm{lt}}(\varpi^r)\left(\mathrm{dg}(\varepsilon,1),1\right) = (\mathrm{lt}(\varepsilon\varpi^r),(\varpi^r,\varepsilon)_n(\varepsilon,\varpi^r)_n) = \widetilde{\mathrm{lt}}(\varepsilon\varpi^r),$$

where ε is a fixed non-square and $1 \le r < l$. It is not difficult to see that other cosets of $\mathcal{A}$ remain distinct in $\widetilde{K'}^l$. ∎

Next, we count the dimension of $\mathcal{H}'$ using a method similar to the one we used for $\mathcal{H}$. We will need the following technical lemma.

Lemma 4.3.6. *Let $\mathbf{b} = \left(\begin{smallmatrix} s & n \\ 0 & t\end{smallmatrix}\right)$ and $\mathbf{b}' = \left(\begin{smallmatrix} s' & n' \\ 0 & t'\end{smallmatrix}\right)$, where $t,t',s,s' \in \mathcal{O}^\times/1+\mathfrak{p}^l$, $n,n' \in \mathcal{O}/\mathfrak{p}^l$ and such that $ss' \in 1+\mathfrak{p}$. Then*

$$\beta'(\mathbf{b},\mathrm{lt}(\varpi^r))\beta'(\mathbf{b}\,\mathrm{lt}(\varpi^r),\mathbf{b}') = 1. \tag{4.3.3}$$

Proof: We use the cocycle formula given in (3.1.5). Note that $\det(\mathrm{lt}(\varpi^r)) = 1$, and hence $v\left(\det(\mathrm{lt}(\varpi^r)), a(\mathbf{b})\mathbf{b}\right) = 1$. Moreover, $a(\mathbf{b})\,\mathbf{b} = \left(\begin{smallmatrix} s & n \\ 0 & s^{-1}\end{smallmatrix}\right)$. Therefore,

$$\beta'\left(\mathbf{b},\mathrm{lt}(\varpi^r)\right) = \beta\left(\left(\begin{matrix} s & n \\ 0 & s^{-1}\end{matrix}\right),\left(\begin{matrix} 1 & 0 \\ \varpi^r & 1\end{matrix}\right)\right) = (s,\varpi^r)_n.$$

To calculate $\beta'(\mathbf{b}\mathrm{lt}(\varpi^r),\mathbf{b}')$, observe that $\det(\mathbf{b}') = s't'$ and $a(\mathbf{b})\mathbf{b}\mathrm{lt}(\varpi^r) = \left(\begin{smallmatrix} s+n\varpi^r & n \\ s^{-1}\varpi^r & s^{-1}\end{smallmatrix}\right)$. Hence, $v\left(\det(\mathbf{b}'), a(\mathbf{b})\mathbf{b}\,\mathrm{lt}(\varpi^r)\right) = (s't',s^{-1})_n = 1$, because s,t and s^{-1} are in $\mathcal{O}^\times$. Moreover, observe that ${}^{a(\mathbf{b}')}\left(a(\mathbf{b})\mathbf{b}\,\mathrm{lt}(\varpi^r)\right)$ is equal to

$$\left(\begin{matrix} 1 & 0 \\ 0 & s'^{-1}t'^{-1}\end{matrix}\right)\left(\begin{matrix} s+n\varpi^r & n \\ s^{-1}\varpi^r & s^{-1}\end{matrix}\right)\left(\begin{matrix} 1 & 0 \\ 0 & s't'\end{matrix}\right) = \left(\begin{matrix} s+n\varpi^r & s't'n \\ s^{-1}s'^{-1}t'^{-1}\varpi^r & s^{-1}\end{matrix}\right),$$

and

$$a(\mathbf{b}')\mathbf{b}' = \left(\begin{matrix} s' & n \\ 0 & s'^{-1}\end{matrix}\right).$$

So, we compute

$$\beta'(\mathbf{b}\,\mathrm{lt}(\varpi^r),\mathbf{b}') = \beta\left(\left(\begin{matrix} s+n\varpi^r & s't'n \\ s^{-1}s'^{-1}t'^{-1}\varpi^r & s^{-1}\end{matrix}\right),\left(\begin{matrix} s' & n \\ 0 & s'^{-1}\end{matrix}\right)\right)$$

$$= (s', s's^{-1}t'^{-1}\varpi^r)_n$$
$$= (s', \varpi^r)_n.$$

Therefore, $\beta'(\mathbf{b}, \mathrm{lt}(\varpi^r))\beta'(\mathbf{b}\ \mathrm{lt}(\varpi^r), \mathbf{b}') = (s, \varpi^r)_n(s', \varpi^r)_n = (ss', \varpi^r)_n = 1$, since $ss' \in 1 + \mathfrak{p}$. ∎

Remark 4.3.7. Observe that $\widetilde{T}^l$ is a subgroup of $\widetilde{T'^l}$. In particular, $T^l \subset \widetilde{T}^l$ is a subgroup of $\widetilde{T'^l}$.

Proposition 4.3.8. *Let $l \geq m$. The dimension of $\mathcal{H}'$ is*

$$\dim\mathcal{H}' = \begin{cases} 1 + (l - m), & \textit{if} \quad \chi'_{i,j}\big|_{T\cap K} \neq 1; \\ 2 + (l - m), & \textit{otherwise.} \end{cases}$$

Proof: To count the basis elements of $\mathcal{H}'$, we determine which double cosets support a function in $\mathcal{H}'$. To do so, recall that for every $f \in \mathcal{H}'$, $\mathtt{b}, \mathtt{b}' \in \widetilde{B'}^l$ and $\mathtt{k} \in \widetilde{K'}^l$, $f(\mathtt{bkb}') = \bar{\chi}'_{ij}(\mathtt{b})f(\mathtt{k})\bar{\chi'}_{i,j}(\mathtt{b}')$. Hence, for every double coset representative $\mathtt{x}$ in Lemma 4.3.5, there exists a function $f \in \mathcal{H}'$ with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{bxb}' = \mathtt{x}$ implies that $\bar{\chi}'_{i,j}(\mathtt{b})\bar{\chi}'_{i,j}(\mathtt{b}') = 1$ for all $\mathtt{b}, \mathtt{b}' \in \widetilde{B'}^l$. Let

$$\mathtt{b} = (\mathbf{b}, \zeta) = \left(\begin{pmatrix} s & n \\ 0 & t \end{pmatrix}, \zeta \right) \quad \text{and} \quad \mathtt{b}' = (\mathbf{b}', \zeta') = \left(\begin{pmatrix} s' & n' \\ 0 & t' \end{pmatrix}, \zeta' \right),$$

where $t, s, t', s' \in \mathcal{O}^\times/1 + \mathfrak{p}^l$, $n, n' \in \mathcal{O}/\mathfrak{p}^l$, be arbitrary elements of $\widetilde{B'}^l$.

The identity coset $\widetilde{B'}^l$: A function $f \in \mathcal{H}'$ has support on $\widetilde{B'}^l$ if and only if $f(\mathtt{b}) = \bar{\chi}'_{i,j}(\mathtt{b})\ \forall \mathtt{b} \in \widetilde{B'}^l$. So there is always a function with support on the identity coset, namely $f = \bar{\chi}'_{i,j}$.

The coset of $\widetilde{w}$: For $\mathtt{b}, \mathtt{b}' \in \widetilde{B'}^l$, $\mathtt{b}\widetilde{w}\mathtt{b}' = \widetilde{w}$ implies that

$$\begin{pmatrix} -ns' & -nn' + st' \\ -ts' & -tn' \end{pmatrix} = w, \tag{4.3.4}$$

and

$$\beta'(\mathbf{b}, w)\beta'(\mathbf{b}w, \mathbf{b}')\zeta\zeta' = 1. \tag{4.3.5}$$

The equation (4.3.4) implies that $n = n' = 0 \mod \mathfrak{p}^l$, $s' = t^{-1} \mod 1 + \mathfrak{p}^l$ and $t' = s^{-1} \mod 1+\mathfrak{p}^l$. Hence $\mathbf{b} = \mathrm{dg}(s,t)$ and $\mathbf{b}' = \mathrm{dg}(t^{-1}, s^{-1}) \mod \widetilde{B'} \cap K'_l$. To compute (4.3.5), observe that by the formula for the 2-cocycle β' given in (3.1.5), we have

$$\beta'(\mathbf{b}, w) = \beta(\mathrm{dg}(s, s^{-1}), w)v(1, a(\mathbf{b})\mathbf{b}) = (1, s^{-1})_n = 1,$$

and

$$\begin{aligned} \beta'(\mathbf{b}w, \mathbf{b}') &= \beta(\begin{pmatrix} n & -ss't' \\ s^{-1}s'^{-1}t'^{-1} & 0 \end{pmatrix}, \begin{pmatrix} s' & n' \\ 0 & s'^{-1} \end{pmatrix})\, v(s't', \begin{pmatrix} n & -s \\ s^{-1} & 0 \end{pmatrix}) \\ &= (s', -s's^{-1}t'^{-1})_n = 1 \end{aligned}$$

since $s', s, t' \in \mathcal{O}^\times$. Therefore, (4.3.5) implies that $\zeta' = \zeta^{-1}$. Note that

$$\chi'_{i,j}\left(\mathrm{dg}(s,t), \zeta\right)\chi'_{i,j}\left(\mathrm{dg}(t^{-1}, s^{-1}), \zeta^{-1}\right) = \chi'_{i,j}(\mathrm{dg}(st^{-1}, ts^{-1}), 1).$$

Note that $\{\mathrm{dg}(st^{-1}, ts^{-1}) \mid t, s \in \mathcal{O}^\times/1 + \mathfrak{p}^l\} \cong T^l$ and as we saw in Remark 4.3.7, T^l is a subgroup of $\widetilde{T'}^l$. Hence, a function $f \in \mathcal{H}'$ has support on $\widetilde{B'}^l\widetilde{w}\widetilde{B'}^l$ if and only if $\chi'_{i,j}\big|_{T^l} = 1$.

The coset of $\widetilde{\mathrm{lt}}(\varpi^r)$:

For $\mathtt{b}$ and $\mathtt{b}'$ in $\widetilde{B'}^l$, suppose $\mathtt{b}\,\widetilde{\mathrm{lt}}(\varpi^r)\mathtt{b}' = \widetilde{\mathrm{lt}}(\varpi^r)$. Then, $\mathtt{b}\,\widetilde{\mathrm{lt}}(\varpi^r) = \widetilde{\mathrm{lt}}(\varpi^r)\mathtt{b}'^{-1}$, which implies

$$\begin{pmatrix} s + n\varpi^r & n \\ t\varpi^r & t \end{pmatrix} = s'^{-1}t'^{-1}\begin{pmatrix} t' & -n' \\ t'\varpi^r & s' - \varpi^r n' \end{pmatrix} \tag{4.3.6}$$

and

$$\beta'(\mathbf{b}, \mathrm{lt}(\varpi^r))\beta'(\mathbf{b}\,\mathrm{lt}(\varpi^r), \mathbf{b}')\zeta\zeta' = 1. \tag{4.3.7}$$

The equation (4.3.6) implies that

$$s + n\varpi^r = s'^{-1} \mod \mathfrak{p}^l \text{ and } t = t'^{-1} - \varpi^r n' s'^{-1} t'^{-1} \mod \mathfrak{p}^l,$$

that is, $ss' \in 1 + \mathfrak{p}^r$, $tt' \in 1 + \mathfrak{p}^r$. By Lemma 4.3.6, $\zeta\zeta' = 1$. Observe that

$$\bar{\chi}'_{i,j}(\mathtt{bb}') = \bar{\chi}'_{i,j}(\begin{pmatrix} ss' & sn' + nt' \\ 0 & tt' \end{pmatrix}, 1),$$

where, $ss', tt' \in 1 + \mathfrak{p}^r$ and $sn' + nt' \in \mathfrak{p}^r$. Therefore, $\bar{\chi}'_{i,j}(\mathtt{bb}') = 1$ if and only if $B' \cap K'_r \subseteq \ker(\chi'_{i,j})$. This latter holds if and only if $r \geq m$, since $\chi'_{i,j}$ is primitive mod m.

Now, let us summarize the result. There is always one function in $\mathcal{H}'$ with support on the identity coset. Also, there are functions in $\mathcal{H}'$ with supports on each of the double cosets represented by $\widetilde{\mathrm{lt}}(\varpi^r)$ for $m \leq r < l$. If $\bar{\chi}'_{i,j}\big|_{T^l} \neq 1$, then $\mathcal{H}'$ does not contain any function with support on the double coset of $\widetilde{w}$; otherwise, there exists an additional function in $\mathcal{H}'$ with support on this double coset. Finally, observe that $\bar{\chi}'_{i,j}|_{T^l} = 1$ if and only if $\chi'_{i,j}|_{T\cap K} = 1$. ∎

Next, we elaborate on the condition $\chi'_{i,j}|_{T\cap K} = 1$ that appears in Proposition 4.3.8.

Lemma 4.3.9. *For $0 \leq i, j < n$, $\chi'_{i,j}|_{T\cap K} = 1$ if and only if $\chi'_{0,0}|_{T\cap K} = \epsilon \circ \vartheta^{j-i}_{\mathcal{O}^\times}$.*

Proof: By Lemma 4.3.1, $\chi'_{i,j}(\iota(a)) = \chi'_{0,0}(\mathrm{dg}(a), \vartheta^{i-j}(a))$, which is equal to 1 if and only if $\chi'_{0,0}|_{T\cap K} = \epsilon \circ \vartheta^{j-i}_{\mathcal{O}^\times}$. ∎

Corollary 4.3.10. *Let $0 \le i, j < n$. The representation $(\mathrm{Ind}_{\widetilde{B'\cap K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_m}$ is irreducible if $\chi'_{0,0}|_{T\cap K} \neq \vartheta_{\mathcal{O}^\times}^{j-i}$, and decomposes into two inequivalent constituents otherwise. Moreover, the quotients $\widetilde{W'}_{i,j,l} := (\mathrm{Ind}_{\widetilde{B'\cap K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_l}/(\mathrm{Ind}_{\widetilde{B'\cap K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_{l-1}}$ for $l > m$ are irreducible.*

Proof: This follows from Lemma 4.3.4, Proposition 1.1.38 and Proposition 4.3.8. ∎

Corollary 4.3.11. *We can decompose $\mathrm{Res}_{\widetilde{K'}}(\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}} \rho')$ as follows:*

$$\mathrm{Res}_{\widetilde{K'}}(\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}} \rho') \simeq \bigoplus_{i,j=0}^{n-1} \left((\mathrm{Ind}_{\widetilde{B'\cap K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_m} \oplus \bigoplus_{l>m} \widetilde{W'}_{i,j,l} \right).$$

If $\chi'_{0,0}|_{T\cap K} \neq \vartheta_{\mathcal{O}^\times}^{k}$, for all $0 \le k < n$, then all the pieces are irreducible. Otherwise, there are exactly n pairs (i, j), $0 \le i, j < n$, such that $j - i \equiv k \mod n$, and $(\mathrm{Ind}_{\widetilde{B'\cap K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_m}$ decomposes into two irreducible constituents. The rest of the constituents are irreducible.

Proof: The result follows from Corollary 4.3.10 and from noting that the map $(i, j) \to j - i \mod n$ has a kernel of size n. Hence, if there exists a pair such that $\chi'_{0,0}|_{T\cap K} = \vartheta_{\mathcal{O}^\times}^{j-i}$, then there are exactly n distinct such pairs. ∎

4.4 Restriction of $\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}} \rho'$ to $\widetilde{K}$

Fix a genuine irreducible representation ρ of $\widetilde{T}$ with central character χ, and let χ_k, $0 \le k < \underline{n}$ be all possible extensions of χ to A, given in Lemma 4.2.1. In Section 4.2, we decomposed $\mathrm{Res}_{\widetilde{K}} \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho$ into irreducible representations of $\widetilde{K}$; this decomposition is given in Corollary 4.2.13. Recall from Corollary 4.2.12 that the irreducible $\widetilde{K}$-spaces $\widetilde{W}_{k,l}^{+}$ and $\widetilde{W}_{k,l}^{-}$, $l \ge m$ that appear in the decomposition in Corollary 4.2.13

are the two irreducible constituents of quotient spaces $\widetilde{W}_{k,l}$, $0 \leq k < \underline{n}$. Moreover, it is part of the statement of Corollary 4.2.12 that, for a fixed k, $\widetilde{W}^+_{k,l}$ and $\widetilde{W}^-_{k,l}$ are inequivalent.

In this section, we show that, for each $0 \leq k < \underline{n}$, $\widetilde{W}_{k,l} \cong \mathrm{Res}_{\widetilde{K}} W'$, where W' is some irreducible representation of $\widetilde{K'}$. We deduce that $\widetilde{W}^+_{k,l}$ is mapped bijectively to $\widetilde{W}^-_{k,l}$ via conjugation by some element in $\widetilde{K'} \setminus \widetilde{K}$. Hence, $\widetilde{W}^+_{k,l}$ and $\widetilde{W}^-_{k,l}$ have the same dimension. This argument allows us to complete the statement of our main result, which is stated in Theorem 4.4.19.

Suppose ρ' is a genuine irreducible representation of $\widetilde{T'}$ with central character χ', such that ρ appears in $\mathrm{Res}_{\widetilde{T}}\rho'$, and let $\chi'_{i,j}$, $0 \leq i, j < n$ be all possible extensions of χ' to A', defined in Lemma 4.3.1. To find a representation W' of $\widetilde{K'}$ satisfying $\widetilde{W}_{k,l} \cong \mathrm{Res}_{\widetilde{K}} W'$, we consider the restriction of the principal series representation $\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$ to $\widetilde{K}$. Because the structure of $\widetilde{T}$ depends on the parity of n, we consider the cases for even and odd n separately. As the first step in each case, we describe the central character χ' of ρ' in relation to χ.

4.4.1 Odd n

Recall that for odd n,

$$\begin{aligned}
Z(\widetilde{T}) &= \{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}, \\
A &= \{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^{\times}, \zeta \in \mu_n,\ n|\mathrm{val}(a)\}, \\
\widetilde{T} \cap \widetilde{K} &= \{(\mathrm{dg}(a), \zeta) \mid a \in \mathcal{O}^{\times}, \zeta \in \mu_n\}, \\
Z(\widetilde{T'}) &= \{(\mathrm{dg}(a,b), \zeta) \mid a, b \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}, \\
A' &= \{(\mathrm{dg}(a,b), \zeta) \mid a, b \in \mathbb{F}^{\times}, \zeta \in \mu_n, n|\mathrm{val}(a), n|\mathrm{val}(b)\}, \\
\widetilde{T'} \cap \widetilde{K'} &= \{(\mathrm{dg}(a,b), \zeta) \mid a, b \in \mathcal{O}^{\times}, \zeta \in \mu_n\}.
\end{aligned}$$

Hence, we have the inclusion diagram in Figure 4.2.

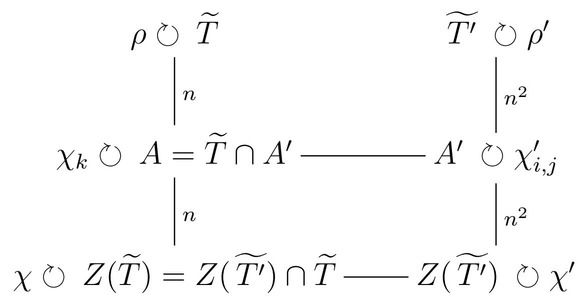


Figure 4.2: Inclusion of subgroups of $\widetilde{T}$ and $\widetilde{T'}$, when n is odd. The representation corresponding to each subgroup is indicated beside ↻. The characters satisfy $\chi_k = \chi'_{i,j}$, where $k \equiv \frac{i-j}{2} \mod n$.

Proposition 4.4.1. *Assume n is odd. Let ρ' be a genuine irreducible representation of $\widetilde{T'}$ such that ρ appears in $\mathrm{Res}_{\widetilde{T}}\rho'$. Then,*

$$\mathrm{Res}_{\widetilde{T}}^{\widetilde{T'}} \rho' \cong \rho^{\oplus n},$$

and $\chi'|_{Z(\widetilde{T})} = \chi$. Conversely, if χ'' is any character of $Z(\widetilde{T'})$, with corresponding irreducible representation ρ'' of $\widetilde{T'}$, satisfying $\chi''|_{Z(\widetilde{T})} = \chi$, then ρ appears in $\mathrm{Res}_{\widetilde{T}}\rho''$.

Proof: We first claim that $X = \{(\mathrm{dg}(1, \varpi^j), 1),\ 0 \le j < n\}$ is a set of double coset representatives of $\widetilde{T} \backslash \widetilde{T'} / A'$. Namely, for every element $\big(\mathrm{dg}(a\varpi^{k_1}, b\varpi^{k_2}), \zeta\big)$ in $\widetilde{T'}$, we can write $k_1 = na_1 + r_1$ for $0 \le r_1 < n$ and $k_2 = na_2 + (j - r_1)$ for $0 \le j < n$. Hence we have

$$\mathrm{dg}(a\varpi^{k_1}, b\varpi^{k_2}) = \mathrm{dg}(\varpi^{r_1}, \varpi^{-r_1})\mathrm{dg}(1, \varpi^j)\mathrm{dg}(a\varpi^{na_1}, b\varpi^{na_2}).$$

Choose $\zeta_1 = 1$ and $\zeta_2 = (b, \varpi^{r_1})_n$. Then every element $\big(\mathrm{dg}(a\varpi^{k_1}, b\varpi^{k_2}), \zeta\big)$ in $\widetilde{T'}$ can be decomposed as $(\mathrm{dg}(\varpi^{r_1}, \varpi^{-r_1}), \zeta_1)\,(\mathrm{dg}(1, \varpi^j), 1)\,(\mathrm{dg}(a\varpi^{na_1}, b\varpi^{na_2}), \zeta_2)$. Therefore, $\widetilde{T'}$ can be decomposed as the disjoint union

$$\widetilde{T'} = \bigsqcup_{0 \le j < n} \widetilde{T}\big(\mathrm{dg}(1, \varpi^j), 1\big) A'.$$

This proves the claim. Therefore, by Mackey's Theorem we have

$$\mathrm{Res}_{\widetilde{T}}\mathrm{Ind}_{A'}^{\widetilde{T'}}\chi_0' \simeq \bigoplus_{\mathtt{x}\in X}\mathrm{Ind}_{\widetilde{T}\cap A'^{\mathtt{x}}}^{\widetilde{T}}{\chi_0'}^{\mathtt{x}}. \tag{4.4.1}$$

Note that $A'^{\mathtt{x}} = A'$. Also, as can be seen in Figure 4.2, $\widetilde{T}\cap A' = A$. So, (4.4.1) is equal to $\bigoplus_{\mathtt{x}\in X}\mathrm{Ind}_A^{\widetilde{T}}{\chi_0'}^{\mathtt{x}}$. On the one hand, because our choice of ρ' implies that ρ appears in the decomposition (4.4.1), $\chi'^{\mathtt{x}}|_{Z(\widetilde{T})} = \chi$ for at least one $\mathtt{x}\in X$. On the other hand, because every $\mathtt{x}\in X$ commutes with $Z(\widetilde{T})$, $\chi'^{\mathtt{x}}|_{Z(\widetilde{T})}$ is the same as $\mathtt{x}$ ranges over X. Hence, the Stone-von Neumann theorem implies that $\mathrm{Ind}_A^{\widetilde{T}}{\chi_0'}^{\mathtt{x}}\cong\rho$ for all $\mathtt{x}\in X$ and hence, $\mathrm{Res}_{\widetilde{T}}^{\widetilde{T'}}\rho'\cong\rho^{\oplus n}$.

Now assume that χ'' is a character of $Z(\widetilde{T'})$ such that $\mathrm{Res}_{Z(\widetilde{T})}\chi'' = \chi$. Then as above $\mathrm{Res}_{\widetilde{T}}\rho''\cong\bigoplus_{\mathtt{x}\in X}\mathrm{Ind}_A^{\widetilde{T}}\chi_0''^{\mathtt{x}}$. For $\mathtt{x} = (\mathrm{I}_2, 1)$, $\mathrm{Res}_{Z(\widetilde{T})}\chi_0''^{\mathtt{x}} = \mathrm{Res}_{Z(\widetilde{T})}\chi_0'' = \chi$, and hence $\mathrm{Ind}_A^{\widetilde{T}}\chi_0''\cong\rho$. ∎

Proposition 4.4.2. *Assume n is odd. Let ρ and ρ' be as in Proposition 4.4.1. Then*

$$\mathrm{Res}_{\widetilde{G}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho' \cong (\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho)^{\oplus n}.$$

Proof: Since $\widetilde{G'} = \widetilde{G}\widetilde{B'}$ and $\widetilde{G}\cap\widetilde{B'} = \widetilde{B}$, Mackey's theorem yields

$$\mathrm{Res}_{\widetilde{G}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho = \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\mathrm{Res}_{\widetilde{B}}\rho',$$

which, by Proposition 4.4.1 equals $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho^{\oplus n} = \left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho\right)^{\oplus n}$. ∎

Proposition 4.4.2 implies that

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho' \cong \mathrm{Res}_{\widetilde{K}}\mathrm{Res}_{\widetilde{G}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho' \cong \left(\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho\right)^{\oplus n}, \tag{4.4.2}$$

and we have computed $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ in Corollary 4.2.13.

Now let us compute $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$ in a different order. Namely, we compute $\mathrm{Res}_{\widetilde{K}}\mathrm{Res}_{\widetilde{K'}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$. We further assume that ρ' is chosen such that depth of χ' is equal to the depth of χ, which equals $m-1$, and that the choice of χ_0 is such that $\mathrm{Res}_A \chi_0' = \chi_0$. By Corollary 4.3.11, we have

$$\mathrm{Res}_{\widetilde{K'}}(\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}} \rho') \simeq \bigoplus_{i,j=0}^{n-1} \left[(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_m} \oplus \bigoplus_{l>m} \widetilde{W'}_{i,j,l} \right], \tag{4.4.3}$$

where $\widetilde{W'}_{i,j,l} = (\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_l} / (\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}} {\chi'}_{i,j})^{K'_{l-1}}$, and the $\chi'_{i,j}$ are defined in Lemma 4.3.1.

In order to study the restriction of each piece in (4.4.3), we need to restrict the characters $\chi'_{i,j}$ to $\widetilde{T} \cap \widetilde{K}$.

Lemma 4.4.3. *Assume n is odd, so that 2 is invertible modulo n. For $0 \le i, j < n$, let k be the integer in $\{0, \cdots, n-1\}$ such that $k \equiv \frac{i-j}{2} \mod n$. Then $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}} \chi'_{i,j} = \chi_k$.*

Proof: Suppose $2k \equiv i - j \mod n$. Let $(\mathrm{dg}(u), \zeta) \in \widetilde{T} \cap \widetilde{K}$. By Lemma 4.3.1,

$$\chi'_{i,j}\left(\mathrm{dg}(u), \zeta\right) = \chi'_0\left(\mathrm{dg}(u), \vartheta(u)^{i-j}\zeta\right),$$

which by Lemma 4.2.1, and because $\chi'_0|_A = \chi_0$, is equal to $\chi_0\left(\mathrm{dg}(u), \vartheta(u)^{2k}\zeta\right) = \chi_k\left(\mathrm{dg}(u), \zeta\right)$. ∎

The cardinality of the kernel of the map $(i, j) \to k \mod n$, in Lemma 4.4.3, is n; that is for each k, there are exactly n distinct characters $\chi'_{i,j}$ of $\widetilde{T'} \cap \widetilde{K'}$ that restrict to χ_k on $\widetilde{T} \cap \widetilde{K}$.

Lemma 4.4.4. *Assume n is odd. Let i, j and k be in $\{0, \cdots, n-1\}$ such that $\chi'_{i,j}|_{\widetilde{T}\cap\widetilde{K}} = \chi_k$. Then, for all $l \ge m$*

$$\mathrm{Res}_{\widetilde{K}} \left(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}} \chi'_{i,j} \right)^{K'_l} \cong \left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}} \chi_k \right)^{K_l}.$$

Proof: By Lemma 4.3.4, it is enough to show that $\mathrm{Res}_{\widetilde{K}^l} \mathrm{Ind}_{\widetilde{B'}^l}^{\widetilde{K'}^l} \bar{\chi'}_{i,j} \cong \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l} \bar{\chi}_k$. Note that $\widetilde{K}^l \backslash \widetilde{K'}^l / \widetilde{B'}^l$ is trivial and $\widetilde{B'}^l \cap \widetilde{K}^l = \widetilde{B}^l$. So, by Mackey's theorem we have

$$\mathrm{Res}_{\widetilde{K}^l} \mathrm{Ind}_{\widetilde{B'}^l}^{\widetilde{K'}^l} \bar{\chi'}_{i,j} \cong \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l} \mathrm{Res}_{\widetilde{B}^l} \bar{\chi'}_{i,j},$$

which is equal to $\mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l} \bar{\chi}_k$ by choice of i, j and k. ∎

Proposition 4.4.5. *Assume n is odd. Let ρ and ρ' be irreducible representations of $\widetilde{T}$ and $\widetilde{T'}$ with central characters χ and χ' respectively, such that $\chi'_0|_A = \chi_0$, where χ_0 and χ'_0 are fixed extensions of χ and χ' to A and A' respectively. Moreover, let $l > m$, and let $\widetilde{W}^-_{k,l} \oplus \widetilde{W}^+_{k,l}$, $0 \leq k < n$, and $\widetilde{W'}_{i,j,l}$, $0 \leq i, j < n$, be the quotient spaces that appear in the decompositions in Corollary 4.2.13 and Corollary 4.3.11 respectively. Then, whenever $k \equiv \frac{i-j}{2} \mod n$, we have*

$$\widetilde{W}^-_{k,l} \oplus \widetilde{W}^+_{k,l} = \mathrm{Res}_{\widetilde{K}} \widetilde{W}'_{i,j,l}.$$

Proof: Recall that $\widetilde{W}'_{i,j,l} = (\mathrm{Ind}_{\widetilde{B'} \cap \widetilde{K'}}^{\widetilde{K'}} \chi'_{i,j})^{K'_l} / (\mathrm{Ind}_{\widetilde{B'} \cap \widetilde{K'}}^{\widetilde{K'}} \chi'_{i,j})^{K'_{l-1}}$. Hence

$$\mathrm{Res}_{\widetilde{K}} \widetilde{W}'_{i,j,l} = \mathrm{Res}_{\widetilde{K}} \left[(\mathrm{Ind}_{\widetilde{B'} \cap \widetilde{K'}}^{\widetilde{K'}} \chi'_{i,j})^{K'_l} \right] / \mathrm{Res}_{\widetilde{K}} \left[(\mathrm{Ind}_{\widetilde{B'} \cap \widetilde{K'}}^{\widetilde{K'}} \chi'_{i,j})^{K'_{l-1}} \right],$$

which by Lemma 4.4.4 is equal to

$$\left(\mathrm{Ind}_{\widetilde{B} \cap \widetilde{K}}^{\widetilde{K}} \chi_k \right)^{K_l} / \left(\mathrm{Ind}_{\widetilde{B} \cap \widetilde{K}}^{\widetilde{K}} \chi_k \right)^{K_{l-1}} = \widetilde{W}^-_{k,l} \oplus \widetilde{W}^+_{k,l}.$$

∎

Remark 4.4.6. The map $\{\widetilde{W}'_{i,j,l} \mid 0 \leq i, j < n\} \xrightarrow{\mathrm{Res}_{\widetilde{K}}} \{\widetilde{W}_{k,l} \mid 0 \leq k < n\}$ is an n to 1 map. That implies each irreducible piece in the decomposition in Corollary 4.2.13 appears n times in $\mathrm{Res}_{\widetilde{K}} \mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}} \rho'$. This is consistent with our result in (4.4.2).

Before continuing to our result in Section 4.4.3, we treat the case of even n.

4.4.2 Even n

Recall that for even n,

$$\begin{aligned} \widetilde{T} &= \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^\times, \zeta \in \mu_n\}, \\ Z(\widetilde{T}) &= \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times n/2}, \zeta \in \mu_n\}, \end{aligned}$$

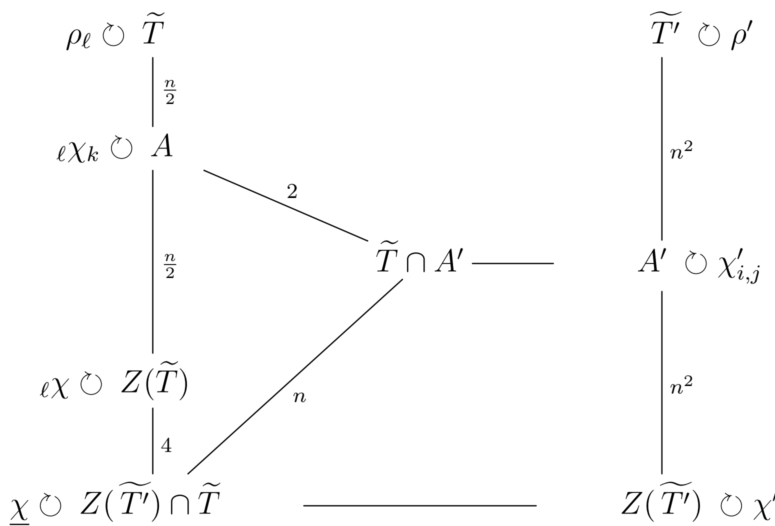


Figure 4.3: Inclusion diagram for subgroups of $\widetilde{T}$ and $\widetilde{T'}$. Symbols on the arrows are the indices. The representation of each subgroup is indicated beside the $\circlearrowright$.

$$\begin{aligned}
A &= \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^\times, \tfrac{n}{2} | \mathrm{val}(t), \zeta \in \mu_n\}, \\
\widetilde{T} \cap \widetilde{K} &= \{(\mathrm{dg}(t), \zeta) \mid t \in \mathcal{O}^\times, \zeta \in \mu_n\}, \\
\widetilde{T'} &= \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in \mathbb{F}^\times, \zeta \in \mu_n\}, \\
Z(\widetilde{T'}) &= \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in {\mathbb{F}^\times}^n, \zeta \in \mu_n\}, \\
A' &= \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in \mathbb{F}^\times, n | \mathrm{val}(t), n | \mathrm{val}(s), \zeta \in \mu_n\}, \\
\widetilde{T'} \cap \widetilde{K} &= \{(\mathrm{dg}(t, s), \zeta) \mid t, s \in \mathcal{O}^\times, \zeta \in \mu_n\},
\end{aligned}$$

and therefore,

$$\begin{aligned}
Z(\widetilde{T'}) \cap \widetilde{T} &= \{(\mathrm{dg}(t), \zeta) \mid t \in {\mathbb{F}^\times}^n, \zeta \in \mu_n\}, \\
A' \cap \widetilde{T} &= \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^\times, n | \mathrm{val}(t), \zeta \in \mu_n\}.
\end{aligned}$$

Figure 4.3 is an inclusion diagram of the above groups. Arrows on the diagram indicate the inclusion, and the symbols on the arrows are the indices. Unlike the case for odd n, the centre $Z(\widetilde{T'})$ and the maximal abelian subgroup A' of $\widetilde{T'}$ do not restrict to those of $\widetilde{T}$ upon restriction to $\widetilde{T}$. Observe that $[Z(\widetilde{T}) : Z(\widetilde{T'}) \cap \widetilde{T}] = 4$, $[A : A' \cap \widetilde{T}] = 2$. This mismatch makes the computation of $\mathrm{Res}_{\widetilde{K}} \mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}} \rho'$ more delicate. We show that, unlike Proposition 4.4.1, our assumption that ρ appears in $\mathrm{Res}_{\widetilde{T}} \rho'$ does not imply that ρ' is ρ isotypic, upon restriction to $\widetilde{T}$. We show instead that ρ is one of the four distinct irreducible representations of $\widetilde{T}$ that appear in $\mathrm{Res}_{\widetilde{T}} \rho'$.

Set $\underline{\chi} := \mathrm{Res}_{Z(\widetilde{T'}) \cap \widetilde{T}} \chi'$. Let $\{e, o\}$ and $\{\mathtt{1}, \mathtt{2}\}$ parametrize sets of coset representatives for $\frac{n}{2}\mathbb{Z}/n\mathbb{Z}$ and $\mathcal{O}^{\times \frac{n}{2}}/\mathcal{O}^{\times n}$ respectively. Define the subgroup

$$Y := \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times \frac{n}{2}},\ n|\mathrm{val}(t)\},$$

of $Z(\widetilde{T})$. Then $Y/\left(Z(\widetilde{T'}) \cap \widetilde{T}\right) \cong \mathcal{O}^{\times \frac{n}{2}}/\mathcal{O}^{\times n}$. By Lemma 1.1.43, $\mathrm{Ind}_{Z(\widetilde{T'}) \cap \widetilde{T}}^{Y} \underline{\chi}$ decomposes into 2 distinct characters of Y: ${}_{\mathtt{1}}\chi$ and ${}_{\mathtt{2}}\chi$. Moreover, $Z(\widetilde{T})/Y \cong \frac{n}{2}\mathbb{Z}/n\mathbb{Z}$. Therefore, for each $\alpha \in \{\mathtt{1}, \mathtt{2}\}$, $\mathrm{Ind}_{Y}^{Z(\widetilde{T})} {}_{\alpha}\chi$ decomposes into two distinct characters of $Z(\widetilde{T})$: ${}_{\alpha e}\chi$ and ${}_{\alpha o}\chi$. Set $L = \{\mathtt{1}e, \mathtt{1}o, \mathtt{2}e, \mathtt{2}o\}$. Hence, $\mathrm{Ind}_{Z(\widetilde{T'}) \cap \widetilde{T}}^{Z(\widetilde{T})} \underline{\chi}$ decomposes into 4 distinct characters ${}_{\ell}\chi$, $\ell \in L$:

$$\mathrm{Ind}_{Z(\widetilde{T'}) \cap \widetilde{T}}^{Z(\widetilde{T})} \underline{\chi} = \bigoplus_{\ell \in L} {}_{\ell}\chi. \tag{4.4.4}$$

Remark 4.4.7. Later, we will consider the restriction of the ${}_{\ell}\chi$ to $Z(\widetilde{T}) \cap \widetilde{K}$. Observe that, upon this restriction ${}_{\mathtt{1}e}\chi = {}_{\mathtt{1}o}\chi$ and ${}_{\mathtt{2}e}\chi = {}_{\mathtt{2}o}\chi$; but, ${}_{\mathtt{1}e}\chi$ and ${}_{\mathtt{2}e}\chi$ remain distinct.

We denote the irreducible genuine representation of $\widetilde{T}$ with central character ${}_{\ell}\chi$ by ρ_ℓ.

Proposition 4.4.8. *Assume n is even. Let ${}_{\ell}\chi$, $\ell \in L$ be as in* (4.4.4)*. Then*

$$\mathrm{Res}_{\widetilde{T}} \rho' = \bigoplus_{\ell \in L} \left[(\rho_\ell)^{\oplus n/2} \right],$$

where the ρ_ℓ are mutually non-isomorphic.

Proof: Note that $X = \{(\mathrm{dg}(1, \varpi^j), 1) \mid 0 \leq j < n\}$ is a system of coset representatives for $\widetilde{T} \backslash \widetilde{T'}/A'$, and that A' is stable under conjugation by $\mathtt{x} \in X$. Moreover, it follows from the definition of the $\chi'_{i,j}$ in Lemma 4.3.1 that for $\mathtt{x} = (\mathrm{dg}(1, \varpi^j), 1)$, ${\chi'_0}^{\mathtt{x}} = \chi'_{0,j}$. Therefore, by Mackey's theorem,

$$\begin{aligned} \mathrm{Res}_{\widetilde{T}}\rho' &= \bigoplus_{\mathtt{x}\in X} \left(\mathrm{Ind}^{\widetilde{T}}_{(\widetilde{T}\cap A'^{\mathtt{x}})} {\chi'_0}^{\mathtt{x}}\right) \\ &= \bigoplus_{j=0}^{n-1} \mathrm{Ind}^{\widetilde{T}}_{A} \left(\mathrm{Ind}^{A}_{\widetilde{T}\cap A'} \chi'_{0,j}\right). \end{aligned} \tag{4.4.5}$$

Observe that $[A : \widetilde{T} \cap A'] = 2$, with coset representatives $\{e, o\}$. Therefore, for every $0 \leq j < n$, $\mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,j}$ is a 2-dimensional representation of the abelian group A and hence decomposes into a direct sum of two characters: ${}_e\chi'_j \oplus {}_o\chi'_j$.

Next we show that the elements of the set $\{{}_e\chi'_j, {}_o\chi'_j \mid 0 \leq j < n\}$ are distinct. By Lemma 1.1.43, for $0 \leq j < n$, $\mathrm{Res}_{\widetilde{T}\cap A'}\mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,j} \cong \chi'_{0,j} \oplus \chi'_{0,j}$. Suppose $0 \leq i, j < n$. Then by Frobenius reciprocity

$$\begin{aligned} \mathrm{Hom}_A\left(\mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,j}, \mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,i}\right) &= \mathrm{Hom}_{\widetilde{T}\cap A'}\left(\mathrm{Res}_{\widetilde{T}\cap A'}\mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,j}, \chi'_{0,i}\right) \\ &= \mathrm{Hom}_{\widetilde{T}\cap A'}\left(\chi'_{0,j} \oplus \chi'_{0,j}, \chi'_{0,i}\right). \end{aligned}$$

We can easily see that $\chi'_{0,j}$ and $\chi'_{0,i}$ coincide on $\widetilde{T} \cap A'$ if and only if $i = j$. Whence,

$$\dim \mathrm{Hom}_A\left(\mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,j}, \mathrm{Ind}^{A}_{\widetilde{T}\cap A'}\chi'_{0,i}\right) = \begin{cases} 2, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the elements of $\{{}_e\chi'_j, {}_o\chi'_j \mid 0 \leq j < n\}$ are $2n$ distinct characters of A, which, because $[A : Z(\widetilde{T})] = n/2$, implies that they restrict to at least 4 distinct characters upon restriction to $Z(\widetilde{T})$. Moreover, because ρ appears in $\mathrm{Res}_{\widetilde{T}}\rho'$, at least one of these 4 central characters is χ. Observe that, for $0 \leq j < n$, and $\alpha \in \{e, o\}$, $\mathrm{Res}_{Z(\widetilde{T'})\cap\widetilde{T}} \; {}_\alpha\chi'_j = \underline{\chi}$.

Consider

$$\mathrm{Ind}^{A}_{Z(\widetilde{T'})\cap\widetilde{T}}\underline{\chi} = \mathrm{Ind}^{A}_{Z(\widetilde{T})}\mathrm{Ind}^{Z(\widetilde{T})}_{Z(\widetilde{T'})\cap\widetilde{T}}\underline{\chi} = \mathrm{Ind}^{A}_{Z(\widetilde{T})}\bigoplus_{\ell\in L} {}_\ell\chi = \bigoplus_{\ell\in L, 0\leq k<n/2} {}_\ell\chi_k.$$

Observe that, by Lemma 1.1.43, the ${}_\ell\chi_k$ are $2n$ distinct characters that, by Lemma 1.1.42, restrict to $\underline{\chi}$ on $Z(\widetilde{T'}) \cap \widetilde{T}$, and therefore exhaust every such character. Hence, the sets $\{{}_e\chi'_{0,j}, {}_o\chi'_{0,j} \mid 0 \leq j < n\}$ and $\{{}_\ell\chi_k \mid \ell \in L, 0 \leq k < n/2\}$ are equal. In particular, by the discussion that follows (4.4.5) we have

$$\begin{aligned} \mathrm{Res}_{\widetilde{T}}\rho' &\cong \mathrm{Ind}_A^{\widetilde{T}} \bigoplus_{0\leq j<n} {}_e\chi'_j \oplus {}_o\chi'_j \\ &= \mathrm{Ind}_A^{\widetilde{T}} \left(\bigoplus_{\ell\in L, 0\leq k<n/2} {}_\ell\chi_k \right) \\ &\cong \bigoplus_{\ell\in L} \rho_\ell^{\oplus\frac{n}{2}}, \end{aligned}$$

because ${}_\ell\chi_k$ extends ${}_\ell\chi$. Moreover, the ρ_ℓ are mutually non-isomorphic, because the ${}_\ell\chi$ are distinct characters of $Z(\widetilde{T})$. ∎

Proposition 4.4.9. *Assume n is even. Let ρ' be as in Proposition 4.4.8. Then the restriction of $\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$ to $\widetilde{G}$ decomposes into a direct sum of four principal series representations of $\widetilde{G}$ as:*

$$\mathrm{Res}_{\widetilde{G}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho' = \bigoplus_{\ell\in L} \left[(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_\ell)^{\oplus\frac{n}{2}} \right]$$

Proof: By Mackey's theorem and Proposition 4.4.8 we have

$$\mathrm{Res}_{\widetilde{G}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho \cong \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\mathrm{Res}_{\widetilde{B}}\rho' \cong \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \left(\bigoplus_{\ell\in L} \rho_\ell \right)^{\oplus\frac{n}{2}} = \bigoplus_{\ell\in L} \left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_\ell \right)^{\oplus\frac{n}{2}}.$$

∎

Proposition 4.4.9 implies that

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G'}}\rho' \cong \mathrm{Res}_{\widetilde{K}} \left(\mathrm{Res}_{\widetilde{G}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G'}}\rho' \right) \cong \bigoplus_{\ell\in L} \left[\left[\mathrm{Res}_{\widetilde{K}} \left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_\ell \right) \right]^{\oplus\frac{n}{2}} \right], \tag{4.4.6}$$

where $\mathrm{Res}_{\widetilde{K}}\left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_\ell\right)$, by Corollary 4.2.13, is

$$\bigoplus_{k=0}^{\frac{n}{2}-1}\left[(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\,{}_\ell\chi_k)^{K_m}\oplus\bigoplus_{l>m}({}_\ell\widetilde{W}_{k,l}^-\oplus{}_\ell\widetilde{W}_{k,l}^+)\right].$$

We further simplify the above decomposition by showing that the characters $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\,{}_\ell\chi_k$, $\ell\in L$, $0\le k<\frac{n}{2}$ are not distinct. Recall that $L=\{1e,2e,1o,2o\}$.

Proposition 4.4.10. *Assume n is even, and let ${}_\ell\chi_k$, $0\le k<\frac{n}{2}$, $\ell\in L$, be all the possible extensions of ${}_\ell\chi$, where ${}_\ell\chi$ is defined in (4.4.4), to A. Then for each k, $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\,{}_\ell\chi_k=\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\,{}_{\ell'}\chi_k$ if and only if $(\ell,\ell')=(1e,1o)$ or $(\ell,\ell')=(2e,2o)$.*

Proof: For a fixed $\ell\in L$, by Lemma 4.2.1, ${}_\ell\chi_k$ appears exactly once in $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_\ell\chi$ for each $0\le k<\frac{n}{2}$. It is not difficult to see that the double coset $\widetilde{T}\cap\widetilde{K}\backslash A/Z(\widetilde{T})$ is trivial. Mackey's theorem yields

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_\ell\chi\cong\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{K}}^{\widetilde{T}\cap\widetilde{K}}\,{}_\ell\chi.$$

Let $\ell,\ell'\in L$. Hence,

$$\mathrm{Hom}_{\widetilde{T}\cap\widetilde{K}}\left(\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_\ell\chi,\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_{\ell'}\chi\right)\cong\mathrm{Hom}_{\widetilde{T}\cap\widetilde{K}}\left(\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_\ell\chi,\mathrm{Ind}_{Z(\widetilde{T})\cap\widetilde{K}}^{\widetilde{T}\cap\widetilde{K}}\,{}_{\ell'}\chi\right),$$

which by Frobenius reciprocity is

$$\mathrm{Hom}_{Z(\widetilde{T})\cap\widetilde{K}}\left(\mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_\ell\chi,{}_{\ell'}\chi\right). \tag{4.4.7}$$

Note that $|Z(\widetilde{T})\cap\widetilde{K}\backslash A/Z(\widetilde{T})|=|\mathcal{O}^\times/\mathcal{O}^{\times n/2}|$. Because A is abelian, Mackey's theorem yields

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T})}^{A}\,{}_\ell\chi\cong{}_\ell\chi^{\oplus\frac{n}{2}}.$$

Hence, (4.4.7) simplifies to $\mathrm{Hom}_{Z(\widetilde{T})\cap\widetilde{K}}\left({}_\ell\chi^{\oplus\frac{n}{2}},{}_{\ell'}\chi\right)$. Thus, for $0\le k<\frac{n}{2}$, $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\,{}_\ell\chi_k=\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\,{}_{\ell'}\chi_k$ if and only if $\mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K}}\,{}_\ell\chi=\mathrm{Res}_{Z(\widetilde{T})\cap\widetilde{K}}\,{}_{\ell'}\chi$. By Remark 4.4.7, the latter holds if and only if $(\ell,\ell')=(1e,1o)$ or $(\ell,\ell')=(2e,2o)$. ∎

Suppose that $\rho \cong \rho_\ell$ for some $\ell \in L$. Without loss of generality we assume that $\rho \cong \rho_{\alpha e}$, $\alpha \in \{1,2\}$. By Proposition 4.4.10, (4.4.6) simplifies to

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho' \cong \bigoplus_{\ell\in\{1e,2e\}}\left[\mathrm{Res}_{\widetilde{K}}\left(\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_\ell\right)\right]^{\oplus n}. \tag{4.4.8}$$

Now, we decompose $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$ in a different order; that is $\mathrm{Res}_{\widetilde{K}}\mathrm{Res}_{\widetilde{K'}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$. Let χ be the central character of ρ. We assume that χ' and χ have the same depth $m-1$. The decomposition of $\mathrm{Res}_{\widetilde{K'}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'$, by Corollary 4.3.11, is

$$\mathrm{Res}_{\widetilde{K'}}(\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho') \simeq \bigoplus_{i,j=0}^{n-1}\left[(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}{\chi'}_{i,j})^{K'_m} \oplus \bigoplus_{l>m}\widetilde{W'}_{i,j,l}\right], \tag{4.4.9}$$

where the $\chi'_{i,j}$ are defined in Lemma 4.3.1. As in the odd case, first we need to study $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}$.

Lemma 4.4.11. *Let* $\chi'_{i,j}$, $0 \le i,j < n$ *be as in* (4.4.9). *Then*

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}\left(\mathrm{dg}(t),\zeta\right) = \chi'_{0,0}\left(\mathrm{dg}(t),\vartheta(t)^{i-j}\zeta\right),$$

for all $(\mathrm{dg}(t),\zeta) \in \widetilde{T}\cap\widetilde{K}$.

Proof: Let $(\mathrm{dg}(t)\zeta) \in \widetilde{T}\cap\widetilde{K}$, by Lemma 4.3.1,

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}\left(\mathrm{dg}(t),\zeta\right) = \chi'_{0,0}\left(\mathrm{dg}(t),\vartheta(t)^{-j}\vartheta(t)^{i}\zeta\right) = \chi'_{0,0}\left(\mathrm{dg}(t),\vartheta(t)^{i-j}\zeta\right).$$

∎

Therefore, $\{\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j} \mid 0 \le i,j < n\}$ consists of n distinct characters of $\widetilde{T}\cap\widetilde{K}$. In the next two lemmas, we relate these characters to the central characters ${}_\ell\chi$, $\ell \in L$.

Lemma 4.4.12. *Each* $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}$ *appears exactly once in* $\bigoplus_{\ell\in\{1e,2e\},0\le k<\frac{n}{2}} {}_\ell\chi_k|_{\widetilde{T}\cap\widetilde{K}}$.

Proof: Note that

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}_{Z(\widetilde{T'})\cap\widetilde{T}}^{A}\underline{\chi} = \mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\left(\bigoplus_{\ell\in L,0\le k<\frac{n}{2}} {}_\ell\chi_k\right),$$

which by Proposition 4.4.10 is

$$\left(\bigoplus_{\ell\in\{1e,2e\},0\le k<\frac{n}{2}} \mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\,\ell\chi_k\right)^{\oplus 2}.$$

Consider

$$\mathrm{Hom}_{\widetilde{T}\cap\widetilde{K}}\left(\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}, \mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\mathrm{Ind}^{A}_{Z(\widetilde{T'})\cap\widetilde{T}}\underline{\chi}\right). \tag{4.4.10}$$

Observe that $\widetilde{T}\cap\widetilde{K}\backslash A/Z(\widetilde{T'})\cap\widetilde{T}\cong\frac{n}{2}\mathbb{Z}/n\mathbb{Z}$. So, by Mackey's theorem and Frobenius reciprocity, (4.4.10) is

$$\mathrm{Hom}_{\widetilde{T}\cap\widetilde{K}}\left(\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}, \left(\mathrm{Ind}^{\widetilde{T}\cap\widetilde{K}}_{Z(\widetilde{T'})\cap\widetilde{K}}\underline{\chi}\right)^{\oplus 2}\right)\cong\mathrm{Hom}_{Z(\widetilde{T'})\cap\widetilde{K}}\left(\mathrm{Res}_{Z(\widetilde{T'})\cap\widetilde{K}}\chi'_{i,j}, \mathrm{Res}_{Z(\widetilde{T'})\cap\widetilde{K}}\underline{\chi}^{\oplus 2}\right).$$

Because $\mathrm{Res}_{Z(\widetilde{T})}\chi'=\underline{\chi}$, for all $0\le i,j<n$, $\mathrm{Res}_{Z(\widetilde{T'})\cap\widetilde{K}}\chi'_{i,j}=\mathrm{Res}_{Z(\widetilde{T'})\cap\widetilde{K}}\underline{\chi}$, and hence, (4.4.10) is 2-dimensional, from which we deduce that $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}$ appears exactly once in $\bigoplus_{\ell\in\{1e,2e\},0\le k<\underline{n}}\ \ell\chi_k$. ∎

In particular, $\chi'_{0,0}$ appears exactly once in $\bigoplus_{\ell\in\{1e,2e\},0\le k<\underline{n}}\ \ell\chi_k$. A priori, $\chi'_{0,0}$ can be any of the characters in that expression; by possibly reordering the index k, we can assume that $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{0,0}={}_{2e}\chi_0$. Then it follows from Lemma 4.2.1 that ${}_{2e}\chi'_k\left(\mathrm{dg}(t),\zeta\right)=\chi'_{0,0}\left(\mathrm{dg}(t),\vartheta(t)^{2k}\zeta\right)$. So, we can see that $\chi'_{1,0}$, given by $\chi'_{0,0}\left(\mathrm{dg}(t),\vartheta(t)\right)$, does not appear as ${}_{2e}\chi_k$, $0\le k<\frac{n}{2}$. Therefore, it ought to appear as ${}_{1e}\chi_k$ for some k. Again, with possible re-indexing, we assume that $\chi'_{1,0}={}_{1e}\chi_0$. With this setting, we have the following lemma.

Lemma 4.4.13. *Assume n is even. For $0\le i,j<n$, if $i-j \mod n$ is even, set $\alpha=2$ and $k=(i-j\mod n)/2$; if $i-j\mod n$ is odd, set $\alpha=1$ and $k=(i-j-1 \mod n)/2$. Then $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}={}_{\alpha e}\chi_k$.*

Proof: Let $(\mathrm{dg}(t),\zeta)$ be an arbitrary element in $\widetilde{T}\cap\widetilde{K}$. Then by Lemma 4.4.11

$$\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{i,j}\left(\mathrm{dg}(t),\zeta\right)=\chi'_{0,0}\left(\mathrm{dg}(t),\vartheta^{i-j}\zeta\right). \tag{4.4.11}$$

Suppose $i-j \mod n$ is even and $k = (i-j \mod n)/2$. Then, because $\chi'_{0,0} = {}_{2e}\chi_0$, (4.4.13) is equal to

$$ {}_{2e}\chi_0\left(\mathrm{dg}(t), \vartheta^{2k}\zeta\right) = {}_{2e}\chi_k\left(\mathrm{dg}(t), \zeta\right), $$

by definition of ${}_{2e}\chi_k$ in Lemma 4.2.1. Now, suppose $i-j \mod n$ is odd, and $k = (i-j-1 \mod n)/2$. Note that $\vartheta^{i-j} = \vartheta\vartheta^{i-j-1}$. Then (4.4.11) is equal to

$$ \chi'_{0,0}\left(\mathrm{dg}(t), \vartheta(t)\vartheta^{i-j-1}(t)\zeta\right) = \chi'_{1,0}\left(\mathrm{dg}(t), \vartheta^{2k}\zeta\right), $$

which, because $\chi'_{1,0} = {}_{1e}\chi_0$, equals ${}_{1e}\chi_0\left(\mathrm{dg}(t), \vartheta^{2k}(t)\zeta\right) = {}_{1e}\chi_k\left(\mathrm{dg}(t), \zeta\right)$. ∎

Lemma 4.4.14. *Assume n is even. For $0 \le i,j < n$, let k and α be as in Lemma 4.4.13. Then, for all $l \ge m$*

$$ \mathrm{Res}_{\widetilde{K}}\left(\mathrm{Ind}_{\widetilde{B'}\cap\widetilde{K'}}^{\widetilde{K'}}\chi'_{i,j}\right)^{K'_l} \cong \left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}{}_{\alpha e}\chi_k\right)^{K_l}. $$

Proof: By Lemma 4.3.4, it is enough to show that $\mathrm{Res}_{\widetilde{K}^l}\mathrm{Ind}_{\widetilde{B'}^l}^{\widetilde{K'}^l}\bar{\chi}'_{i,j} \cong \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}{}_{\alpha e}\bar{\chi}_k$. Mackey's theorem yields $\mathrm{Res}_{\widetilde{K}^l}\mathrm{Ind}_{\widetilde{B'}^l}^{\widetilde{K'}^l}\bar{\chi}'_{i,j} \cong \mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}\mathrm{Res}_{\widetilde{B}^l}\bar{\chi}'_{i,j}$, which is equal to $\mathrm{Ind}_{\widetilde{B}^l}^{\widetilde{K}^l}{}_{\alpha e}\bar{\chi}_k$ by Lemma 4.4.13. ∎

For each $\ell \in L$ and $0 \le k < \frac{n}{2}$, let ${}_\ell\widetilde{W}_k$ be the quotient space that appears in $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_\ell$ in Corollary 4.2.13.

Proposition 4.4.15. *Assume n is even. Let ρ and ρ' be irreducible representations of $\widetilde{T}$ and $\widetilde{T'}$ with central characters χ and χ' respectively, such that ρ appears in $\mathrm{Res}_{\widetilde{T}}\rho'$, and that χ and χ' are primitive mod m. For $l > m$, $0 \le k < \frac{n}{2}$, $0 \le i,j < n$, let $\widetilde{W}_{k,l} = \widetilde{W}^-_{k,l} \oplus \widetilde{W}^+_{k,l}$ and $\widetilde{W'}_{i,j,l}$ be the quotient spaces that appear in the decompositions in Corollary 4.2.13 and Corollary 4.3.11 respectively. Then, for each $0 \le k < \frac{n}{2}$, $l > m$, $\widetilde{W}_{k,l} = \mathrm{Res}_{\widetilde{K}}\widetilde{W'}_{i,j,l}$, for some $0 \le i,j < n$.*

Proof: Recall that for $0 \le k < \frac{n}{2}$, $\widetilde{W}_{k,l} = (\mathrm{Ind}_{\widetilde{T}\cap\widetilde{K}}^{\widetilde{K}}\chi_k)^{K_l}/(\mathrm{Ind}_{\widetilde{T}\cap\widetilde{K}}^{\widetilde{K}}\chi_k)^{K_{l-1}}$. It follows from Proposition 4.4.8 that $\rho = \rho_\ell$, and hence $\chi = {}_\ell\chi$, for some $\ell \in L$, where

${}_{\ell}\chi$ is defined in (4.4.4). By Proposition 4.4.10, without loss of generality, we can assume that $\ell \in \{1e, 2e\}$. Moreover, by a consequence of Lemma 4.4.12, we can assume that ${}_{\ell}\chi_0$ are chosen such that ${}_{2e}\chi_0 = \mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{0,0}$ and ${}_{1e}\chi_0 = \mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\chi'_{1,0}$.

If $\chi = {}_{2e}\chi$ and $\widetilde{W}_{k,l}$ is given, for some $0 \leq k < \underline{n}$, $l > m$. Consider $\widetilde{W}'_{2k,0,l}$. Then by Lemma 4.4.13 and Lemma 4.4.14, $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\widetilde{W}'_{2k,0,l} = {}_{2e}\widetilde{W}_{k,l} = \widetilde{W}_{k,l}$. If $\chi = {}_{1e}\chi$ and $\widetilde{W}_{k,l}$ is given; consider $\widetilde{W}'_{2k-1,0,l}$. Then by Lemma 4.4.13 and Lemma 4.4.14, $\mathrm{Res}_{\widetilde{T}\cap\widetilde{K}}\widetilde{W}'_{2k-1,0,l} = {}_{1e}\widetilde{W}_{k,l} = \widetilde{W}_{k,l}$. ∎

Remark 4.4.16. Proposition 4.4.15 can be seen directly from Lemma 4.4.12, without identifying the quotient space $\widetilde{W}'_{i,j}$ that restricts to a given $\widetilde{W}_{k,l}$.

Remark 4.4.17. Note that the map $(i,j) \to i-j \mod n$, which appears in Lemma 4.4.11, has a kernel of size n. Therefore, it follows from Lemma 4.4.14 that

$$\mathrm{Res}_{\widetilde{K}} \bigoplus_{0\leq i,j<n} \widetilde{W}'_{i,j,l} \cong \bigoplus_{0\leq k<\frac{n}{2}} \left({}_{2e}\widetilde{W}_{k,l}\right)^{\oplus n} \oplus \bigoplus_{0\leq k<\frac{n}{2}} \left({}_{1e}\widetilde{W}_{k,l}\right)^{\oplus n}.$$

In particular, every irreducible constituent in the decomposition of $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ appears n times in $\mathrm{Res}_{\widetilde{K}}\left(\mathrm{Res}_{\widetilde{K'}}\mathrm{Ind}_{\widetilde{B'}}^{\widetilde{G'}}\rho'\right)$. This is consistent with our result in (4.4.8).

4.4.3 Main Result

Finally, we put all of our results together to make the main theorem of this chapter. We first state the common corollary to Proposition 4.4.5 and Proposition 4.4.15.

Corollary 4.4.18. *Let ρ be a genuine irreducible representation of $\widetilde{T}$ with central character χ. The non-equivalent irreducible representations $\widetilde{W}^-_{k,l}$ and $\widetilde{W}^+_{k,l}$, $0 \leq k < \underline{n}$, $l > m$, that appear in the* K*-type decomposition $\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ in Corollary 4.2.13 are of the same dimension.*

Proof: By Proposition 4.4.5 and Proposition 4.4.15, for any $0 \leq k < \underline{n}$, $l > m$, $\widetilde{W}_{k,l} = \widetilde{W}^-_{k,l} \oplus \widetilde{W}^+_{k,l}$, is restriction of some irreducible representation $\widetilde{W}'_{i,j}$ of $\widetilde{K'}$, for some $0 \leq i,j < n$. Hence, there exists an element of $\widetilde{K'} \setminus \widetilde{K}$ that maps $\widetilde{W}^-_{k,l}$ into $\widetilde{W}^+_{k,l}$ bijectively. ∎

Theorem 4.4.19. *Let ρ be a genuine irreducible representation of $\widetilde{T}$ with central character χ, and let χ_k, $0 \leq k < \underline{n}$, be all the possible extensions of χ to A. Then*

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho \cong \bigoplus_{k=0}^{\underline{n}-1} \left((\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_k)^{K_m} \right) \oplus \bigoplus_{l>m} \left(\widetilde{W}^+_{0,l} \oplus \widetilde{W}^-_{0,l} \right)^{\oplus \underline{n}},$$

where $\widetilde{W}^+_{0,l}$ and $\widetilde{W}^-_{0,l}$ are two inequivalent representations of $\widetilde{K}$ with the same dimension and $\left(\widetilde{W}^+_{0,l} \oplus \widetilde{W}^-_{0,l} \right) \cong (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{K_l} / (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{K_{l-1}}$.

We consider $(T \cap K)^2$ as a subgroup of $\widetilde{T}$. The m-level representations

$$\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i \right)^{K_m},$$

where $0 \leq i < \underline{n}$, are mutually non-isomorphic, except when $m = 1$ and $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta|^{-j}_{\mathcal{O}^{\times 2}}$, for some $0 \leq j < \underline{n}$. In this case

$$\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i \right)^{K_1} \cong \left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_k \right)^{K_1},$$

exactly when $i + k \equiv j \mod n$.

All the pieces in the decomposition are irreducible except when $m = 1$ and $\chi_0|_{(T\cap K)^2} = \epsilon \circ \vartheta|^{-2i}_{\mathcal{O}^{\times 2}}$ for some $0 \leq i < \underline{n}$. In this case, we are in one of the following situations:

1. *If $4 \nmid n$ then there is exactly one $0 \leq i < \underline{n}$ for which $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

2. *If $4|n$ then there are exactly two $0 \leq i,k < \underline{n}$, $|i-k| = \frac{n}{4}$ for which $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{K_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

Proof: The decomposition and irreducibility results follow from Corollary 4.2.12 and Corollary 4.2.13. The multiplicity results are shown in Corollary 4.2.15, and the fact that $\widetilde{W}_{0,l}^{+}$ and $\widetilde{W}_{0,l}^{-}$ have the same degree follows from Corollary 4.4.18. ∎

Chapter 5

The Reducibility Points of the Unramified Principal Series of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$

In this chapter, we define the unramified principal series of $\widetilde{G}$ and determine completely when a representation in this series is reducible. As shown in Chapter 3, the structure of the torus of $\widetilde{G}$ depends on the parity of n. We only study the case for odd n and leave the case of even n for future work. We continue using non-normalized induction.

The problem of finding the reducibility points of the representations of the covering groups was solved for unitary unramified principal series of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$ by Moen in [Moe88]. For non-unitary unramified principal series, the case of $n = 2$ was solved by Gelbart and Sally [GS75], and $n = 3$ was solved by Aritürk [Ari80]. Our approach is mostly aligned with the argument for linear groups, which can be found in [Cas95].

Unramified principal series representations are parametrized by a complex number $\mathbf{s}$ that identifies the character of the centre of the torus $\widetilde{T}$. These characters are either regular or singular, and are defined in Section 5.1.1. The case with a regular central character is addressed in Section 5.2. The main idea is to identify a certain integral operator from the unramified principal series representation to its contragredient representation. With composition of two such operators, we construct an intertwining operator $\mathcal{T}$ from the unramified principal series representation to itself. The key Proposition 5.2.6 relates the question of the irreducibility of the principal series representation to the one of calculating $\mathcal{T}$, which is done by tracing the image of the spherical function, defined in Section 5.1, under $\mathcal{T}$. The main result for this set of representations, proved in Theorem 5.2.13, states that regular unramified principal series representations of $\widetilde{G}$ are reducible exactly when $\mathbf{s} = 1 \pm \frac{1}{n}$. Those principal series representations with a non-regular central character, however, demand a completely different treatment, which is addressed in Section 5.3. The main result for this group, proved in Theorem 5.3.6, is that non-regular unramified principal series representations of $\widetilde{G}$ are reducible exactly when $\mathbf{s} = 1 + \frac{\pi i}{\log q}$.

Throughout this chapter, we assume that n is odd such that $n|q-1$.Recall that we denote the elements of the linear group G by boldface font style, for example $\mathbf{g} \in G$; elements of the covering group $\widetilde{G}$ by typewriter font style, for example $\mathtt{g} \in \widetilde{G}$; and those of the p-adic field $\mathbb{F}$ in roman font style, for example $t \in \mathbb{F}$.

5.1 Spherical Space

5.1.1 Unramified Principal Series

Recall that an unramified character of the multiplicative group $\mathbb{F}^\times$ is a character that is trivial on the maximal compact subgroup $\mathcal{O}^\times$. We extend this notion to genuine characters of $Z(\widetilde{T})$. Recall that $Z(\widetilde{T}) = \{(\mathrm{dg}(t), \zeta) \mid t \in \mathbb{F}^{\times n}, \zeta \in \mu_n\}$.

Definition 5.1.1. *We call a genuine character of $Z(\widetilde{T})$ unramified if it is trivial on the subgroup $\{(\mathrm{dg}(t), 1) \mid t \in \mathcal{O}^{\times n}\} \cong \mathcal{O}^{\times n}$.*

All such characters have the form

$$\begin{aligned} \chi_{\mathbf{s}} : Z(\widetilde{T}) &\to \mathbb{C} \\ (\mathrm{dg}(t), \zeta) &\mapsto |t|^{\mathbf{s}}\epsilon(\zeta), \end{aligned}$$

for some $\mathbf{s} \in \mathbb{C}$, where $|\cdot|$ is the norm character.

For linear groups, *regular characters* are those which are not fixed by any Weyl group element. That definition is consistent with normalized induction; if one uses non-normalized induction, then the regular characters χ are instead those such that $\chi \neq \delta\chi^{w}$. We use non-normalized induction, so the correct analogue of regular characters are those $\chi_{\mathbf{s}}$ for which $\chi_{\mathbf{s}} \neq \delta_{\tilde{B}}\chi_{\mathbf{s}}^{\tilde{w}}$. Otherwise, we call $\chi_{\mathbf{s}}$ a *non-regular* character. Here, $\delta_{\tilde{B}}$ is the restriction of the modular character of the Borel subgroup $\widetilde{B}$ to $Z(\widetilde{T})$, i.e, $\delta_{\tilde{B}}\,(\mathrm{dg}(t),\zeta) = |t|^2$, $t \in \mathbb{F}^{\times n}$.

Lemma 5.1.2. *The non-regular unramified characters of $Z(\widetilde{T})$ are $\chi_{\mathbf{s}}$, for $\mathbf{s} \in \{1, \frac{\pi i}{\log q} + 1\}$.*

Proof: Observe that for $(\mathrm{dg}(t), \zeta) \in Z(\widetilde{T})$,

$$\chi_{\mathbf{s}}^{\tilde{w}}\,(\mathrm{dg}(t),\zeta) = \chi_{\mathbf{s}}\left(\mathrm{dg}(t^{-1}),\zeta\right) = |t|^{-\mathbf{s}}\epsilon(\zeta) = \chi_{-\mathbf{s}}\,(\mathrm{dg}(t),\zeta)\,.$$

Therefore, $\chi_{\mathbf{s}}^{\tilde{w}} = \chi_{-\mathbf{s}}$ and hence $\delta_{\tilde{B}}\chi_{\mathbf{s}}^{\tilde{w}} = \chi_{-\mathbf{s}+2}$. So, $\chi_{\mathbf{s}} = \delta_{\tilde{B}}\chi_{\mathbf{s}}^{\tilde{w}}$ if and only if $|\cdot|^{\mathbf{s}} = |\cdot|^{-\mathbf{s}+2}$; that is $q^{-n\mathbf{s}} = q^{n(\mathbf{s}-2)}$. Therefore, $-n\mathbf{s} = n(\mathbf{s}-2) + \frac{2\pi i k}{\log q}$, $k \in \mathbb{Z}$, which further implies that $s = 1 - \frac{\pi i k}{n \log q}$, $k \in \mathbb{Z}$. Observe that $q^{n\mathbf{s}} = q^{n(\mathbf{s}+\frac{2\pi k i}{n\log q})}$ implies that $\chi_{\mathbf{s}} = \chi_{\mathbf{s}+\frac{2\pi i}{n \log q}}$. We deduce that there are exactly two distinct non-regular unramified characters; one corresponds to the even choices of k, and the other corresponds to the odd choices of k. Because n is odd, we can assume that $k \in \{0, -n\}$, therefore, distinct non-regular unramified characters are determined by $\mathbf{s} \in \{1, 1 + \frac{\pi i}{\log q}\}$. ∎

Recall from Section 3.3 that $A = C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$. For an unramified character $\chi_{\mathbf{s}}$ of $Z(\widetilde{T})$, we choose the trivial extension of $\chi_{\mathbf{s}}$ to A, in the sense that it is trivial on elements $\iota(u)$, $u \in \mathcal{O}^\times$ of A. We continue calling this extension $\chi_{\mathbf{s}}$. So, for $(\mathrm{dg}(a), \zeta) \in A$, we have $\chi_{\mathbf{s}}(\mathrm{dg}(a), \zeta) = |a|^{\mathbf{s}}\epsilon(\zeta)$.

Let $\rho_{\mathbf{s}}$ be the genuine irreducible representation of $\widetilde{T}$ with central character $\chi_{\mathbf{s}}$, and let $\rho_{\mathbf{s}}$ also denote its trivial extension to $\widetilde{B}$. Let $\pi_{\mathbf{s}} = \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$ be the corresponding principal series representation of $\widetilde{G}$. We refer to $\pi_{\mathbf{s}}$ as the *unramified principal series* of $\widetilde{G}$.

5.1.2 A Spherical Function

In the argument for determining conditions for irreducibility of the unramified principal series of G [Cas95], one considers a spherical function that generates the one-dimensional subspace of K-fixed vectors in the principal series. We adapt Casselman's argument [Cas95] to our setting. A first step is to define the correct analogue of the spherical function. Because $\pi_{\mathbf{s}}$ is a genuine representation, $\pi_{\mathbf{s}}^{\widetilde{K}}$ is trivial. Since $\widetilde{G}$ splits over K, we can instead consider the fixed points under the subgroup $\widetilde{K}_0 \cong K$ in $\widetilde{K}$.

Recall that $\widetilde{K}_0 = \{(\mathbf{k}, s(\mathbf{k})^{-1}) \mid \mathbf{k} \in K\}$, and $\widetilde{K} \simeq \widetilde{K}_0 \times \mu_n$. We extend ϵ to a character $\widetilde{\epsilon}$ of $\widetilde{K}$:

$$\begin{aligned} \widetilde{\epsilon} : \widetilde{K} &\rightarrow \mathbb{C} \\ \mathtt{k}(\mathrm{I}_2, \zeta) &\mapsto \epsilon(\zeta), \end{aligned} \tag{5.1.1}$$

for $\mathtt{k} \in \widetilde{K}_0$ and $\zeta \in \mu_n$. Therefore, $\widetilde{K}_0$ acts trivially on a genuine representation of $\widetilde{K}$ if and only if $\widetilde{K}$ acts by $\widetilde{\epsilon}$.

Lemma 5.1.3. *Let $\pi_{\mathbf{s}}$ be an unramified principal series of $\widetilde{G}$. The space $\pi_{\mathbf{s}}^{\widetilde{K}_0}$ of $\widetilde{K}_0$-fixed points of $\pi_{\mathbf{s}}$ is one-dimensional.*

Proof: Note that

$$\pi_{\mathbf{s}}{}^{\widetilde{K}_0} \cong (\mathrm{Res}_{\widetilde{K}}\pi_{\mathbf{s}})^{\widetilde{K}_0}.$$

Also observe that $\chi_{\mathbf{s}}|_{\widetilde{T}\cap\widetilde{K}}(\mathrm{dg}(u),\zeta)=\epsilon(\zeta)$, $u\in\mathcal{O}^\times,\zeta\in\mu_n$. It follows from Proposition 4.2.2 and (4.2.2) that

$$(\mathrm{Res}_{\widetilde{K}}\pi_{\mathbf{s}})^{\widetilde{K}_0} \;\cong\; \bigoplus_{i=0}^{n-1}\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{\widetilde{K}_0},$$

where χ_0 is the trivial extension to $\widetilde{B}\cap\widetilde{K}$ of $\chi_{\mathbf{s}}|_{\widetilde{T}\cap\widetilde{K}}$, and we recall from Lemma 4.2.1 that

$$\chi_i\left((\mathrm{dg}(u),\zeta)\right)=\chi_{\mathbf{s}}\left(\left(\mathrm{dg}(u),\vartheta_{\mathcal{O}^\times}^{2i}(u)\zeta\right)\right)=\epsilon\left(\vartheta_{\mathcal{O}^\times}^{2i}(u)\right)\epsilon\left(\zeta\right),$$

for all $u\in\mathcal{O}^\times$.

It is therefore enough to calculate the dimension of $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{\widetilde{K}_0}$ for each $0\le i<n$. By definition of $\widetilde{\epsilon}$, $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i\right)^{\widetilde{K}_0}$ is the subspace on which $\widetilde{K}$ acts by $\widetilde{\epsilon}$. By Frobenius reciprocity

$$\mathrm{Hom}_{\widetilde{K}}(\widetilde{\epsilon},\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i) \;\cong\; \mathrm{Hom}_{\widetilde{B}\cap\widetilde{K}}(\widetilde{\epsilon}|_{\widetilde{B}\cap\widetilde{K}},\chi_i).$$

Note that for $u\in\mathcal{O}^\times$, $(\mathrm{dg}(u),\zeta)=\iota(u)\,(\mathrm{I}_2,\zeta)$, where $(\mathrm{dg}(u),1)\in\widetilde{K}_0$. Hence, $\widetilde{\epsilon}|_{\widetilde{T}\cap\widetilde{K}}(\mathrm{dg}(u),\zeta)=\epsilon(\zeta)$; that is $\widetilde{\epsilon}|_{\widetilde{T}\cap\widetilde{K}}=\chi_0$. Therefore,

$$\dim\mathrm{Hom}_{\widetilde{T}\cap\widetilde{K}}(\widetilde{\epsilon}|_{\widetilde{T}\cap\widetilde{K}},\chi_i)=\begin{cases}1, & i=0\\ 0, & \text{otherwise.}\end{cases}$$

It follows that $\dim(\pi_{\mathbf{s}})^{\widetilde{K}_0}=\dim\bigoplus_{i=0}^{n-1}(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_i)^{\widetilde{K}_0}=1$. ∎

Observe that for any non-zero function $\phi\in\pi_{\mathbf{s}}^{\widetilde{K}_0}$, and any $\mathtt{g}\in\widetilde{G}$ that is factored as $\mathtt{g}=\mathtt{ntk}$, $\mathtt{n}\in N,\mathtt{t}\in\widetilde{T}$ and $\mathtt{k}\in\widetilde{K}_0$ with respect to the Iwasawa decomposition of $\widetilde{G}$ in Lemma 3.3.8, we have:

$$\phi(\mathtt{ntk})=\phi(\mathtt{nt})=\rho_{\mathbf{s}}(\mathtt{nt})\phi(1)=\rho_{\mathbf{s}}(\mathtt{t})\phi(1).$$

Therefore, ϕ is entirely determined by $\phi(1)$, which is a function in $\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$. We further identify $\phi(1)$ up to a scalar multiple in the next lemma. Let $f_0\in\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$ be the function with support on A with the property that $f_0(1)=1$.

Lemma 5.1.4. *Let ϕ be a non-zero function in $\pi_{\mathbf{s}}^{\widetilde{K}_0}$. Then $\phi(1)$ belongs to the one-dimensional subspace of* $\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$ *generated by f_0.*

Proof: Because $\phi \in \pi_{\mathbf{s}}^{\widetilde{K}_0}$, it is fixed under the action of $\widetilde{T} \cap \widetilde{K}_0$. An arbitrary element of $\widetilde{T} \cap \widetilde{K}_0$ has the form of $\iota(u)$ for some $u \in \mathcal{O}^\times$. Since ϕ is fixed by $\widetilde{K}_0$, $\pi_{\mathbf{s}}(\iota(u))\,\phi = \phi$, thus $(\pi_s(\iota(u))\phi)(1) = \phi(1)$. On the other hand, by right translation, we have $(\pi_{\mathbf{s}}(\iota(u))\phi)(1) = \phi(\iota(u))$; and since $\iota(u)$ is in $\widetilde{T}$, this equals $\rho_{\mathbf{s}}(\iota(u))(\phi(1))$. Thus $\phi(1)$ is fixed by $\rho_{\mathbf{s}}(\iota(u))$. We next show that this implies the support of $\phi(1)$ is A.

Let $\mathtt{t} \in \widetilde{T}$. Observe that $\{\iota(\varpi^i) \mid 0 \leq i < n\}$ is a set of coset representatives for A in $\widetilde{T}$. So $\mathtt{t} \in A\iota(\varpi^j)$ for some $0 \leq j < n$. Then there exists $\mathtt{a} \in A$ such that $\mathtt{t} = \mathtt{a}\iota(\varpi^j)$, whence

$$\mathtt{t}\iota(u) = \mathtt{a}\iota(\varpi^j)\iota(u) = \mathtt{a}\iota(\varpi^j)\iota(u)\iota(\varpi^{-j})\iota(\varpi^j) = \mathtt{a}\mathtt{y}\iota(\varpi^j),$$

where $\mathtt{y} = \iota(\varpi^j)\iota(u)\iota(\varpi^{-j}) = (\mathrm{dg}(u), (u, \varpi^j)_n^2)$ is in $\widetilde{T} \cap \widetilde{K} \subset A$. On the one hand, $\iota(u) \in \widetilde{K}_0$ implies that

$$(\pi_{\mathbf{s}}(\iota(u))\,\phi)\,(1)(\mathtt{t}) = \phi(1)(\mathtt{a}\iota(\varpi^j)) = \chi_{\mathbf{s}}(\mathtt{a})\left(\phi(1)\left(\iota(\varpi^j)\right)\right).$$

On the other hand, $\iota(u) \in \widetilde{T}$ implies that

$$\begin{aligned}
\left(\left(\pi_{\mathbf{s}}(\iota(u))\phi\right)(1)\right)(\mathtt{t}) &= \rho_{\mathbf{s}}(\iota(u))\phi(1)(\mathtt{t}) \\
&= \phi(1)(\mathtt{t}\iota(u)) \\
&= \phi(1)(\mathtt{a}\mathtt{y}\iota(\varpi^j)) \\
&= \chi_{\mathbf{s}}(\mathtt{a})\chi_{\mathbf{s}}(\mathtt{y})\left(\phi(1)\iota(\varpi^j)\right).
\end{aligned}$$

Hence, if $\phi(1)(\iota(\varpi^j)) \neq 0$, then $\chi_{\mathbf{s}}(\mathtt{y}) = 1$ for all $\mathtt{y} \in \widetilde{T} \cap \widetilde{K}$. We evaluate

$$\chi_{\mathbf{s}}(\mathtt{y}) = \chi_{\mathbf{s}}\big(\mathrm{dg}(u), (u, \varpi^j)_n^2\big) = \epsilon\left((u, \varpi^j)_n^2\right) = \vartheta(u)^{-2j}.$$

But, ϑ^{-2j} is trivial if and only if $j = 0$, whence the support of $\phi(1)$ is the identity coset of A. ∎

Lemma 5.1.5. *Let* $\mathtt{u} = (\mathrm{dg}(u), \zeta)$ *be an arbitrary element of* $\widetilde{T} \cap \widetilde{K}$. *Then*

$$\rho_{\mathbf{s}}(\mathtt{u}) f_0 = \epsilon(\zeta) f_0.$$

Proof: Recall that $\mathrm{Supp}(f_0) = A$. It is easy to see that for all $\mathtt{t} = (\mathrm{dg}(t), \delta) \in \widetilde{T}$, $\mathrm{val}(tu) = \mathrm{val}(t)$. Hence, $\mathtt{tu} \in A$ if and only if $\mathtt{t} \in A$. Therefore, because $\rho_{\mathbf{s}}$ acts by right translation, $\mathrm{Supp}(\rho_{\mathbf{s}}(\mathtt{u}) f_0) = \mathrm{Supp}(f_0) = A$. Let $\mathtt{t} \in A$. Since $A = C_{\widetilde{T}}(\widetilde{T} \cap \widetilde{K})$, $\mathtt{t}$ and $\mathtt{u}$ commute. Therefore,

$$\rho_{\mathbf{s}}(\mathtt{u}) f_0(\mathtt{t}) = f_0(\mathtt{tu}) = f_0(\mathtt{ut}) = \chi_{\mathbf{s}}(\mathrm{dg}(u), \zeta) f_0(\mathtt{t}) = \epsilon(\zeta) f_0(\mathtt{t}).$$

∎

For $\mathtt{g} \in \widetilde{G}$ with Iwasawa decomposition $\mathtt{g} = \mathtt{ntk}$, $\mathtt{n} \in N, \mathtt{t} \in \widetilde{T}$ and $\mathtt{k} \in \widetilde{K}_0$, set $\phi_{\mathbf{s}}(\mathtt{g}) = \rho_{\mathbf{s}}(\mathtt{t}) f_0$. We show that this map is well-defined.

Lemma 5.1.6. *The map* $\phi_{\mathbf{s}}$ *is a well-defined element of* $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho_{\mathbf{s}}$.

Proof: If $\phi_{\mathbf{s}}$ is well-defined then by construction it is in $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho_{\mathbf{s}}$. We show that $\phi_{\mathbf{s}}(\mathtt{g})$ is independent of the decomposition of $\mathtt{g}$. Let $\mathtt{g} \in \widetilde{G}$, and let $\mathtt{n}_1\mathtt{t}_1\mathtt{k}_1$ and $\mathtt{n}_2\mathtt{t}_2\mathtt{k}_2$ be two different Iwasawa decompositions of $\mathtt{g}$. We will show that $\rho_{\mathbf{s}}(\mathtt{t}_1) f_0 = \rho_{\mathbf{s}}(\mathtt{t}_2) f_0$. It follows from $\mathtt{n}_1\mathtt{t}_1\mathtt{k}_1 = \mathtt{n}_2\mathtt{t}_2\mathtt{k}_2$ that

$$(\mathtt{n}_2\mathtt{t}_2)^{-1}\mathtt{n}_1\mathtt{t}_1 = \mathtt{k}_2\mathtt{k}_1^{-1}, \tag{5.1.2}$$

which is an element of $(N\widetilde{T}) \cap \widetilde{K}_0$. Let us write the left hand side of (5.1.2) as $\mathtt{n}_3\mathtt{t}_3$. Since $N \cap \widetilde{T} = \{(\mathrm{I}_2, 1)\}$, it is easy to see that $\mathtt{n}_3 \in N \cap \widetilde{K}_0$ and $\mathtt{t}_3 \in \widetilde{T} \cap \widetilde{K}_0$. Hence, from (5.1.2) we have

$$\begin{aligned} \mathtt{t}_1 &= \mathtt{n}_1^{-1}\mathtt{n}_2\mathtt{t}_2\mathtt{n}_3\mathtt{t}_3 \\ &= \mathtt{n}_1^{-1}\mathtt{n}_2\mathtt{t}_2\mathtt{n}_3\mathtt{t}_2^{-1}\mathtt{t}_2\mathtt{t}_3 \end{aligned}$$

$$= \mathtt{n}_1^{-1}\mathtt{n}_2{}^{\mathtt{t}_2}\mathtt{n}_3\mathtt{t}_2\mathtt{t}_3,$$

where ${}^{\mathtt{t}_2}\mathtt{n}_3 \in N$ since $\widetilde{T}$ normalizes N. Therefore, we have

$$\begin{aligned} \rho_{\mathbf{s}}(\mathtt{t}_1)f_0 &= \rho_{\mathbf{s}}(\mathtt{n}_1^{-1}\mathtt{n}_2{}^{\mathtt{t}_2}\mathtt{n}_3\mathtt{t}_2\mathtt{t}_3)f_0 \\ &= \rho_{\mathbf{s}}(\mathtt{t}_2\mathtt{t}_3)f_0. \end{aligned}$$

Observe that $\mathtt{t}_3 \in \widetilde{T} \cap \widetilde{K}_0$, so $\mathtt{t}_3 = \iota(u)$ for some $u \in \mathcal{O}^\times$. Hence, it follows from Lemma 5.1.5 that $\rho_{\mathbf{s}}(\mathtt{t}_3)f_0 = f_0$ and hence $\rho_{\mathbf{s}}(\mathtt{t}_1)f_0 = \rho_{\mathbf{s}}(\mathtt{t}_2)f_0$, as required. ∎

Since in particular $\phi_{\mathbf{s}}(\mathrm{I}_2, 1) = f_0$, it follows from the preceding lemma that $\phi_{\mathbf{s}}$ is a basis for ${\pi_{\mathbf{s}}}^{\widetilde{K}_0}$. We call $\phi_{\mathbf{s}}$ the *normalized spherical function* of $\pi_{\mathbf{s}}$.

5.1.3 Quasi-Characteristic Function of $\widetilde{K}$

Consider the space $C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$, on which $\widetilde{G}$ acts by right-translation. Recall that functions in this space are locally constant with compact support, so they are integrable. In this section, we define an intertwining map $P_{\mathbf{s}} : C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}) \to \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$; and we identify a pre-image of $\phi_{\mathbf{s}}$ under this map, which we call the *quasi-characteristic* function of $\widetilde{K}$.

We assume the Haar measure $\mu_{\widetilde{B}} = \mu$ on $\widetilde{B}$ is right-invariant, recall $\delta_{\widetilde{B}}\left(\left(\begin{smallmatrix} t & n \\ 0 & t^{-1}\end{smallmatrix}\right), \zeta\right) = |t|^2$. For every $\varphi \in C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ define $P_{\mathbf{s}}(\varphi) : \widetilde{G} \to \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$ by

$$\begin{aligned} P_{\mathbf{s}}(\varphi)(\mathtt{g}) : \widetilde{T} &\to \mathbb{C} \\ \mathtt{t} &\to \int_{\widetilde{B}} \rho_{\mathbf{s}}(\mathtt{b}^{-1})\varphi(\mathtt{bg})(\mathtt{t})d\mathtt{b}. \end{aligned} \tag{5.1.3}$$

It is not difficult to see that $P_{\mathbf{s}}(\varphi)$ is smooth. Moreover, because $\rho_{\mathbf{s}}(\mathtt{b}^{-1})\varphi(\mathtt{bg}) \in \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$ for all $\mathtt{b} \in \widetilde{B}$ and $\mathtt{g} \in \widetilde{G}$, $P_{\mathbf{s}}(\varphi)(\mathtt{g})(\mathtt{at}) = \chi_{\mathbf{s}}(\mathtt{a})P_{\mathbf{s}}(\varphi)(\mathtt{g})(\mathtt{t})$ for every $\mathtt{a} \in A$ and $\mathtt{t} \in \widetilde{T}$. So, $P_{\mathbf{s}}(\varphi)(\mathtt{g}) \in \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$.

The following lemma shows that the image of $P_{\mathbf{s}}$ lies in the space of the representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$. Evidently, $P_{\mathbf{s}}(\mathtt{g}\cdot\varphi) = \mathtt{g}\cdot P_{\mathbf{s}}(\varphi)$, so $P_{\mathbf{s}}$ defines an intertwining map

$$P_{\mathbf{s}} : C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}) \to \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}.$$

Lemma 5.1.7. *For every* $\varphi \in C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$*,* $P_{\mathbf{s}}(\varphi)$ *lies in* $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$*. Moreover,* $P_{\mathbf{s}}$ *takes* $\widetilde{K}_0$*-fixed vectors in* $C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ *to* $\widetilde{K}_0$*-fixed vectors in* $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$*.*

Proof: For the first claim, we need to show that $P_{\mathbf{s}}(\varphi)(\mathtt{b}_1\mathtt{g}) = \rho_{\mathbf{s}}(\mathtt{b}_1)P(\varphi)(\mathtt{g})$ for all $\mathtt{b}_1 \in \widetilde{B}$ and $\mathtt{g} \in \widetilde{G}$. We compute

$$\begin{aligned} P_{\mathbf{s}}(\varphi)(\mathtt{b}_1\mathtt{g}) &= \int_{\widetilde{B}} \rho_{\mathbf{s}}(\mathtt{b}^{-1})\varphi(\mathtt{b}\mathtt{b}_1\mathtt{g})d\mathtt{b} \\ &= \int_{\widetilde{B}} \rho_{\mathbf{s}}(\mathtt{b}_1\mathtt{b}'^{-1})\varphi(\mathtt{b}'\mathtt{g})d\mathtt{b}'\mathtt{b}_1^{-1} \qquad \text{where } \mathtt{b}' = \mathtt{b}\mathtt{b}_1 \\ &= \rho_{\mathbf{s}}(\mathtt{b}_1)\int_{\widetilde{B}} \rho_{\mathbf{s}}(\mathtt{b}'^{-1})\varphi(\mathtt{b}'\mathtt{g})d\mathtt{b}' \\ &= \rho_{\mathbf{s}}(\mathtt{b}_1)P_{\mathbf{s}}(\varphi)(\mathtt{g}). \end{aligned}$$

So $P_{\mathbf{s}}(\varphi)$ lies in $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$. Moreover, suppose $\varphi \in C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ is such that for all $\mathtt{k} \in \widetilde{K}_0$, $\mathtt{k}\cdot\varphi(\mathtt{g}) = \varphi(\mathtt{g})$. Since $P_{\mathbf{s}}$ is an intertwining operator for the action of $\widetilde{G}$, $P_{\mathbf{s}}(\varphi) = P_{\mathbf{s}}(\mathtt{k}\cdot\varphi) = \mathtt{k}\cdot P_{\mathbf{s}}(\varphi)$, so $P_{\mathbf{s}}(\varphi) \in \pi_{\mathbf{s}}^{\widetilde{K}_0}$. ∎

Now we wish to identify a simple function in the pre-image under $P_{\mathbf{s}}$ of our normalized spherical function $\phi_{\mathbf{s}}$. In fact, Lemma 5.1.7 implies that $P_{\mathbf{s}}$ maps any $\widetilde{K}_0$-fixed function to a scalar multiple of $\phi_{\mathbf{s}}$, so it suffices to choose a simple function with this property. Set $M = \mu(\widetilde{B} \cap \widetilde{K})$, and define the function $\varphi_0 \in C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ via

$$\varphi_0(\mathtt{g}) = \begin{cases} \widetilde{\epsilon}(\mathtt{g})M^{-1}f_0, & \text{if } \mathtt{g} \in \widetilde{K} \\ 0, & \text{if } \mathtt{g} \notin \widetilde{K}, \end{cases} \tag{5.1.4}$$

where $\widetilde{\epsilon}$ is defined in (5.1.1). We call the function φ_0 the *quasi-characteristic function* of $\widetilde{K}$. Note that φ_0 is $\widetilde{K}_0$-fixed under right translation.

Lemma 5.1.8. *Let* $P_{\mathbf{s}}$ *be as defined in* (5.1.3)*,* $\phi_{\mathbf{s}}$ *be the normalized spherical function, and* φ_0 *be the quasi-characteristic function. Then* $P_{\mathbf{s}}(\varphi_0) = \phi_{\mathbf{s}}$*.*

Proof: By construction, we have $\mathtt{k}\cdot\varphi_0 = \varphi_0$ for all $k \in \widetilde{K}_0$, so by Lemma 5.1.7, $P_{\mathbf{s}}(\varphi_0) \in \pi_{\mathbf{s}}^{\widetilde{K}_0}$. This space is one-dimensional, so $P_{\mathbf{s}}(\varphi_0) = \lambda\phi_{\mathbf{s}}$ for some scalar λ. To show that $\lambda = 1$, it is enough to show that $P_{\mathbf{s}}(\varphi_0)(\mathrm{I}_2, 1) = f_0$. Note that since φ_0 is zero outside of $\widetilde{K}$ we have

$$P_{\mathbf{s}}(\varphi_0)(\mathrm{I}_2, 1) \;=\; \int_{\widetilde{B}\cap\widetilde{K}} \rho_{\mathbf{s}}(\mathtt{b}^{-1})\varphi_0(\mathtt{b})d\mathtt{b}.$$

Fix $\mathtt{b} = \mathtt{k}(\mathrm{I}_2, \zeta) \in \widetilde{B}\cap\widetilde{K}$, for some $\mathtt{k} \in \widetilde{B}\cap\widetilde{K}_0$ and $\zeta \in \mu_n$. By (5.1.4), $\varphi_0(\mathtt{b}) = \epsilon(\zeta)M^{-1}f_0$. Now, because $\mathtt{b} \in N\widetilde{T}\cap\widetilde{K}$, and because $\rho_{\mathbf{s}}$ is trivial on the unipotent radical subgroup N, Lemma 5.1.5 implies that

$$\rho_{\mathbf{s}}(\mathtt{b}^{-1})f_0 \;=\; \epsilon(\zeta)^{-1}f_0.$$

Therefore, $\rho_{\mathbf{s}}(\mathtt{b}^{-1})\varphi_0(\mathtt{b}) = \epsilon(\zeta)^{-1}\epsilon(\zeta)M^{-1}f_0 = M^{-1}f_0$ and

$$\int_{\widetilde{B}\cap\widetilde{K}} \rho_{\mathbf{s}}(\mathtt{b}^{-1})\varphi_0(\mathtt{b})d\mathtt{b} = M^{-1}f_0\mu(\widetilde{B}\cap\widetilde{K}) = f_0.$$

∎

5.1.4 Jacquet Module

In this section, we set the stage in order to identify an intertwining map $\mathcal{T}_{\mathbf{s}}$ from $\pi_{\mathbf{s}}$ to $\pi_{-\mathbf{s}+2}$. Let $(\pi_{\mathbf{s}})_N = \pi_{\mathbf{s}}/\pi_{\mathbf{s}}(N)$ be the Jacquet module of $\pi_{\mathbf{s}}$, defined in Definition 1.1.39. Our goal in this section is to construct an exact sequence

$$0 \to \rho_{-\mathbf{s}+2} \to (\pi_{\mathbf{s}})_N \to \rho_{\mathbf{s}} \to 0.$$

Later, in Section 5.2, we specify to those parameters $\mathbf{s}$ that yield a regular central character. But for now, we include all the possible values of $\mathbf{s}$.

Consider the $\widetilde{B}$-map

$$\alpha : \pi_{\mathbf{s}} \;\to\; \rho_{\mathbf{s}}$$

$$f \mapsto f(1).$$

Since α is non-zero and $\rho_{\mathbf{s}}$ is irreducible, α is surjective. Let $\bar{\alpha}$ be the image of α under the Jacquet functor. Observe that $\rho_{\mathbf{s}}|_N$ is trivial; and hence, $(\rho_{\mathbf{s}})_N$, the Jacquet module of $\rho_{\mathbf{s}}$, is $\rho_{\mathbf{s}}$. Because the Jacquet functor is exact, $\bar{\alpha} : (\pi_{\mathbf{s}})_N \to \rho_{\mathbf{s}}$ is a surjective $\widetilde{T}$-map, so $(\pi_{\mathbf{s}})_N / \ker(\bar{\alpha}) \cong \rho_{\mathbf{s}}$. In general, it is a well-known result, originally due to Jacquet [Jac71], that the Jacquet module of a subrepresentation of a principal series representation is non-zero.

Next, we identify $\rho_{-\mathbf{s}+2}$ as a subspace of $(\pi_{\mathbf{s}})_N$. To do so, we define an intertwining operator from a $\widetilde{B}$-subspace of $\pi_{\mathbf{s}}$, which consists of those functions with support on $\widetilde{B}\widetilde{w}\widetilde{B}$, to $\rho_{-\mathbf{s}+2}$ and that factors through to an isomorphism under the Jacquet functor.

Set $\mathrm{In}_{\widetilde{w}} = \{h \in \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}} \mid \mathrm{Supp}(h) \subseteq \widetilde{B}\widetilde{w}\widetilde{B}\}$ and $\pi_{\widetilde{w}} := \pi_{\mathbf{s}}|_{\widetilde{B}}$. Then $(\pi_{\widetilde{w}}, \mathrm{In}_{\widetilde{w}})$ is a $\widetilde{B}$-subspace of $(\pi_{\mathbf{s}}, \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}})$. Observe that, because $\widetilde{G} = \widetilde{B} \cup \widetilde{B}\widetilde{w}\widetilde{B}$, $h \in \mathrm{In}_{\widetilde{w}}$ if and only if $h(1) = 0$, i.e., $h \in \ker(\alpha)$. Hence, $\mathrm{In}_{\widetilde{w}} = \ker(\alpha)$. For any representation (π, V), recall that $\mathcal{E}(V)$ denotes the vector space underlying (π, V).

We will benefit from the following decomposition of an element $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta)$ in $\widetilde{G}$ when $c \neq 0$:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta) = (\mathrm{ut}(ac^{-1}), 1)(\mathrm{dg}(-c^{-1}), \zeta)\, \widetilde{w}\, (\mathrm{ut}(dc^{-1}), 1). \tag{5.1.5}$$

Lemma 5.1.9. *Let $h \in \mathrm{In}_{\widetilde{w}}$. Then $h|_{\widetilde{w}N}$ is compactly supported and therefore,*

$$\int_N h(\widetilde{w}\mathrm{n})d\,\mathtt{n},$$

is convergent.

Proof: By assumption $\mathrm{Supp}(h) \subseteq \widetilde{B}\widetilde{w}\widetilde{B} = \widetilde{B}\widetilde{w}N$. We will show that there exists an open compact subgroup N_0 of N such that $\mathrm{Supp}(h) \subseteq \widetilde{B}\widetilde{w}N_0$.

We have $\mathrm{Supp}(h) \subseteq \widetilde{B}\widetilde{w}\widetilde{B}$ if and only if $h(\mathrm{I}_2, 1) = 0$. Because h is locally constant, $h(\mathrm{I}_2, 1) = 0$ implies that there exists an integer $k > 0$ and a compact open subgroup $\bar{N}_k = \{(\mathrm{lt}(x), \zeta) \mid x \in \mathfrak{p}^k\}$ such that $h|_{\bar{N}_k} = 0$. If $x \neq 0$, by (5.1.5) we have

$$(\mathrm{lt}(x), \zeta) = \left(\mathrm{ut}(x^{-1}), 1\right) \left(\mathrm{dg}(-x^{-1}), \zeta\right) \widetilde{w} \left(\mathrm{ut}(x^{-1}), 1\right).$$

Hence,

$$h\left(\mathrm{lt}(x), \zeta\right) = \rho_{\mathbf{s}}\left(\mathrm{dg}(-x^{-1}), \zeta\right) h\left(\widetilde{w}\left(\mathrm{ut}(x^{-1}), 1\right)\right). \tag{5.1.6}$$

If $x \in \mathfrak{p}^k$ then the left-hand side of 5.1.6 is zero. Moreover, $x \in \mathfrak{p}^k$ implies that $\mathrm{val}(x^{-1}) \leq -k$, whence h is zero on $\{\widetilde{w}\,(\mathrm{ut}(x), \zeta) \mid \mathrm{val}(x) \leq -k\}$. Therefore, $\mathrm{Supp}(h) \subset \widetilde{B}\widetilde{w}N_{-k+1}$, where $N_{-k+1} := \{(\mathrm{ut}(x), \zeta) \mid x \in \mathfrak{p}^{-k+1}\}$. ∎

Remark 5.1.10. The integral in Lemma 5.1.9 can be written as an infinite sum of integrals over compact sets, which turns out to be a geometric series that is convergent when $\mathrm{Re}(\mathbf{s}) > 1$. It follows that in that case, the integral in Lemma 5.1.9 is convergent for all $h \in \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$.

If $h \in \mathrm{In}_{\widetilde{w}}$, then $\int_N h(\widetilde{w}\mathtt{n})d\,\mathtt{n}$ is convergent by Lemma 5.1.9, and assumes values in $\mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$. Consider the map

$$\begin{aligned} \Lambda'_{\widetilde{w},\mathbf{s}} : \mathrm{In}_{\widetilde{w}} &\to \mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}) \\ h &\mapsto \int_N h(\widetilde{w}\mathtt{n})d\,\mathtt{n}. \end{aligned} \tag{5.1.7}$$

The map $\Lambda'_{\widetilde{w},\mathbf{s}}$ induces an action of $\widetilde{T}$ on $\mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ that is different from the natural action on this space.

Lemma 5.1.11. *The map $\Lambda'_{\widetilde{w},\mathbf{s}}$ is a $\widetilde{T}$-intertwining operator from $\pi_{\widetilde{w}}$ to $\delta_{\widetilde{B}}\rho_{\mathbf{s}}^{\widetilde{w}}$.*

Proof: It is easy to see that for $f \in \mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ and $(\mathrm{dg}(t), \zeta) \in \widetilde{T}$,

$$\delta_{\widetilde{B}}\rho_{\mathbf{s}}^{\widetilde{w}}\left(\mathrm{dg}(t), \zeta\right) f = |t|^2 \rho_{\mathbf{s}}\left(\mathrm{dg}(t)^{-1}, \zeta\right) f$$

defines an action of $\widetilde{T}$ on $\mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$. Let $h \in \mathrm{In}_{\widetilde{w}}$ and $\mathtt{t} \in \widetilde{T}$. Then

$$\Lambda'_{\widetilde{w},\mathbf{s}}(\mathtt{t}\cdot h) = \int_N (\mathtt{t}\cdot h)(\widetilde{w}\mathtt{n})d\mathtt{n}. \tag{5.1.8}$$

Because $\pi_{\widetilde{w}}$ is the restriction of $\pi_{\mathbf{s}}$, $\widetilde{T}$ acts by right translation. Therefore,

$$(\mathtt{t}\cdot h)(\widetilde{w}\mathtt{n}) = h(\widetilde{w}\mathtt{n}\mathtt{t}) = h(\widetilde{w}\mathtt{t}\mathtt{t}^{-1}\mathtt{n}\mathtt{t}).$$

Because $\widetilde{T}$ normalizes N, $\mathtt{n}' = \mathtt{t}^{-1}\mathtt{n}\mathtt{t}$ is in N. It is not difficult to see that $d\,\mathtt{n} = d\,\mathtt{t}\mathtt{n}'\mathtt{t}^{-1} = \delta_{\tilde{B}}(\mathtt{t})d\,\mathtt{n}'$. Hence, by this change of variable, (5.1.8) is equal to

$$\delta_{\tilde{B}}(\mathtt{t})\int_N h(\widetilde{w}\mathtt{t}\mathtt{n})d\mathtt{n}.$$

Observe that, because functions in $\mathrm{In}_{\widetilde{w}} \subset \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}}$ translate on the left by $\rho_{\mathbf{s}}$, we have

$$h(\widetilde{w}\mathtt{t}\mathtt{n}) = h(\widetilde{w}\mathtt{t}\widetilde{w}^{-1}\widetilde{w}\mathtt{n}) = \rho_{\mathbf{s}}(\widetilde{w}\mathtt{t}\widetilde{w}^{-1})h(\widetilde{w}\mathtt{n}) = \rho_{\mathbf{s}}^{\widetilde{w}}(\mathtt{t})h(\widetilde{w}\mathtt{n}).$$

Hence,

$$\Lambda'_{\widetilde{w},\mathbf{s}}(\mathtt{t}\cdot h) = \delta_{\tilde{B}}(\mathtt{t})\rho_{\mathbf{s}}^{\widetilde{w}}(\mathtt{t})\int_N h(\widetilde{w}\mathtt{n})d\mathtt{n} = \delta_{\tilde{B}}(\mathtt{t})\rho_{\mathbf{s}}^{\widetilde{w}}(\mathtt{t})\Lambda'_{\widetilde{w},\mathbf{s}}(h).$$

∎

The next lemma shows that $\Lambda'_{\widetilde{w},\mathbf{s}}$ defines, indeed, an intertwining operator from $(\pi_{\widetilde{w}}, \mathrm{In}_{\widetilde{w}})$ to $(\rho_{-\mathbf{s}+2}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2})$.

Lemma 5.1.12. *The two $\widetilde{T}$-representations $\delta_{\tilde{B}}\rho_{\mathbf{s}}^{\widetilde{w}}$ and $\rho_{-\mathbf{s}+2}$ are isomorphic.*

Proof: Since $\rho_{-\mathbf{s}+2}$ and $\delta_{\tilde{B}}\rho_{\mathbf{s}}^{\widetilde{w}}$ are irreducible genuine representations of $\widetilde{T}$, by the Stone-von Neumann theroem, it suffices to show that they have the same central character. Recall that the central character of $\rho_{-\mathbf{s}+2}$ is $\chi_{-\mathbf{s}+2}$. Let $(\mathrm{dg}(t), \zeta) \in Z(\widetilde{T})$. We compute

$$\delta_{\tilde{B}}\rho_{\mathbf{s}}^{\widetilde{w}}(\mathrm{dg}(t),\zeta) = |t|^2\rho_{\mathbf{s}}\left(\mathrm{dg}(t)^{-1},\zeta\right) = |t|^2|t|^{-\mathbf{s}}\epsilon(\zeta) = |t|^{-\mathbf{s}+2}\epsilon(\zeta) = \chi_{-\mathbf{s}+2}(\mathrm{dg}(t),\zeta),$$

and hence the result. ∎

We identify $(\delta_{\tilde{B}}\rho_{\mathbf{s}}^{\widetilde{w}}, \mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}))$ with $(\rho_{-\mathbf{s}+2}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2})$, hence $\Lambda'_{\widetilde{w},\mathbf{s}}$ defines an intertwining map $\Lambda_{\widetilde{w},\mathbf{s}} : \mathrm{In}_{\widetilde{w}} \to \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2}$.

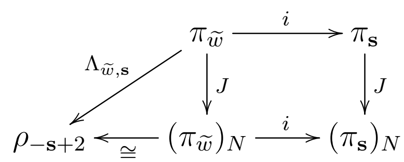


Figure 5.1: The image of $\Lambda_{\widetilde{w},\mathbf{s}}$ under the Jacquet functor J is an isomorphism. The inclusion map is denoted by i. The Jacquet functor is exact, and the diagram commutes.

Lemma 5.1.13. *The homomorphism $\Lambda_{\widetilde{w},\mathbf{s}}$ factors through to a $\widetilde{T}$-isomorphism between $((\pi_{\widetilde{w}})_N, (\mathrm{In}_{\widetilde{w}})_N)$ and $(\rho_{-\mathbf{s}+2}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2})$.*

Proof: Since $(\rho_{-\mathbf{s}+2}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2})$ is irreducible, the non-zero homomorphism $\Lambda_{\widetilde{w},\mathbf{s}}$ is surjective. We show that $\ker(\Lambda_{\widetilde{w},\mathbf{s}})$ is the subspace $\mathrm{In}_{\widetilde{w}}(N)$ of $\mathrm{In}_{\widetilde{w}}$, and hence $\mathrm{In}_{\widetilde{w}N} = \mathrm{In}_{\widetilde{w}}/\mathrm{In}_{\widetilde{w}}(N) \cong \mathrm{Img}(\Lambda_{\widetilde{w},\mathbf{s}}) = \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2}$.

Suppose $h \in \ker(\Lambda_{\widetilde{w},\mathbf{s}})$. By Lemma 1.1.40, to show that $h \in \mathrm{In}_{\widetilde{w}}(N)$, we need to show that there exists a compact open subgroup N_0 of N such that

$$\int_{N_0} \pi_{\mathbf{s}}(\mathtt{n})h d\mathtt{n} = 0.$$

Observe that, by the proof of Lemma 5.1.9, $\mathrm{Supp}(h) \subseteq \widetilde{B}\widetilde{w}N'$ for some open compact subgroup N' of N. Therefore,

$$\Lambda_{\widetilde{w},\mathbf{s}}(h) = \int_{N'} h(\widetilde{w}\mathtt{n})d\mathtt{n} = \int_{N'} \pi_{\mathbf{s}}(\mathtt{n})h(\widetilde{w})d\mathtt{n} = 0.$$

Let $\mathtt{g} \in \widetilde{G} = \widetilde{B} \cup \widetilde{B}\widetilde{w}N$. If $\mathtt{g} \in \widetilde{B}$, then $h(\mathtt{g}) = 0$, so clearly $\int_{N'} \pi_{\mathbf{s}}(\mathtt{n})h(\mathtt{g})d\mathtt{n} = 0$. Otherwise, $\mathtt{g} = \mathtt{b}\widetilde{w}\mathtt{n}'$ for $\mathtt{b} \in \widetilde{B}, \mathtt{n}' \in N$ and

$$\begin{aligned}
\int_{N'} \pi_{\mathbf{s}}(\mathtt{n})h(\mathtt{b}\widetilde{w}\mathtt{n}')d\mathtt{n} &= \rho_{\mathbf{s}}(\mathtt{b})\int_{N'} \pi_{\mathbf{s}}(\mathtt{n}'\mathtt{n})h(\widetilde{w})d\mathtt{n} \\
&= \rho_{\mathbf{s}}(\mathtt{b})\int_{N'} \pi_{\mathbf{s}}(\mathtt{n}'')h(\widetilde{w})d\mathtt{n}'^{-1}\mathtt{n}'' \qquad \mathtt{n}'\mathtt{n} = \mathtt{n}'' \\
&= \rho_{\mathbf{s}}(\mathtt{b})\int_{N'} \pi_{\mathbf{s}}(\mathtt{n}'')h(\widetilde{w})d\mathtt{n}'' = 0.
\end{aligned}$$

Therefore,

$$\int_{N'} \pi_{\mathbf{s}}(\mathtt{n})h\, d\mathtt{n} = 0,$$

and hence $h \in \mathrm{In}_{\widetilde{w}}(N)$. Conversely, assume $h \in \mathrm{In}_{\widetilde{w}}(N)$. Then $h = \sum_{i=1}^{m}(\pi_{\widetilde{w}}(\mathtt{n}_i)h_i - h_i)$ for some $m \in \mathbb{Z}$, $h_i \in \mathrm{In}_{\widetilde{w}}$ and $\mathtt{n}_i \in N$. Let N_0 be a compact subgroup of N such that $\mathrm{Supp}(h_i) \subset \widetilde{B}\widetilde{w}N_0$, and that it contains n_i for all $1 \leq i \leq m$. Then

$$\begin{aligned} \int_N h(\widetilde{w}\mathtt{n})d\mathtt{n} &= \sum_{i=1}^{m} \int_{N_0} \left(\pi_{\widetilde{w}}(\mathtt{n}_i)h_i - h_i\right)(\widetilde{w}\mathtt{n})d\mathtt{n} \\ &= \sum_{i=1}^{m} \left(\int_{N_0} h_i(\widetilde{w}\mathtt{n}\mathtt{n}_i)d\mathtt{n} - \int_{N_0} h_i(\widetilde{w}\mathtt{n})d\mathtt{n} \right). \end{aligned} \tag{5.1.9}$$

Since $d\mathtt{n}$ is invariant, with a change of variable $\mathtt{n}\mathtt{n}_i = \mathtt{n}'$, for each i

$$\int_{N_0} h_i(\widetilde{w}\mathtt{n}\mathtt{n}_i)d\mathtt{n} = \int_{N_0} h_i(\widetilde{w}\mathtt{n}')d\mathtt{n}',$$

and hence (5.1.9) is equal to zero. Therefore, $h \in \ker(\Lambda_{\widetilde{w},\mathbf{s}})$. So, $\ker(\Lambda_{\widetilde{w},\mathbf{s}}) = \mathrm{In}_{\widetilde{w}}(N)$, whence $((\pi_{\widetilde{w}})_N, \mathrm{In}_{\widetilde{w}N}) \cong \left(\rho_{-\mathbf{s}+2}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2}\right)$. ∎

We denote the isomorphism given in Lemma 5.1.13 by $\bar{\Lambda}_{\widetilde{w},\mathbf{s}}$. As shown in Figure 5.1, via $\bar{\Lambda}_{\widetilde{w},\mathbf{s}}$, and the inclusion map $(\pi_{\widetilde{w}})_N \hookrightarrow (\pi_{\mathbf{s}})_N$, we identify $\rho_{-\mathbf{s}+2}$ as a subrepresentation of $(\pi_{\mathbf{s}})_N$. Consider the exact sequence,

$$0 \to \pi_{\widetilde{w}} \xrightarrow{i} \pi_{\mathbf{s}} \xrightarrow{\alpha} \rho_{\mathbf{s}} \to 0, \tag{5.1.10}$$

where $\alpha(f) = f(1)$. Because the Jacquet functor J is exact, the image of (5.1.10) under J is an exact sequence. Moreover, by Lemma 5.1.13, $(\pi_{\widetilde{w}})_N \cong \rho_{-\mathbf{s}+2}$. Hence,

$$0 \to \rho_{-\mathbf{s}+2} \xrightarrow{i} (\pi_{\mathbf{s}})_N \xrightarrow{\bar{\alpha}} \rho_{\mathbf{s}} \to 0, \tag{5.1.11}$$

where $\bar{\alpha} = J(\alpha)$, is an exact sequence.

5.2 The Case of Regular Characters

In this section, we show that if $\chi_{\mathbf{s}}$ is a regular character, so that $\rho_{\mathbf{s}} \neq \rho_{-\mathbf{s}+2}$, then the exact sequence (5.1.11) splits; that is, $(\pi_{\mathbf{s}})_N \cong \rho_{-\mathbf{s}+2} \oplus \rho_{\mathbf{s}}$. We then show that this decomposition leads to constructing a homomorphism $\mathcal{T}_{\mathbf{s}} \in \mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{-\mathbf{s}+2})$.

Lemma 5.2.1. *Suppose $\chi_{\mathbf{s}}$ is a regular character. Then the sequence*

$$0 \to \rho_{-\mathbf{s}+2} \xrightarrow{i} (\pi_{\mathbf{s}})_N \xrightarrow{\bar{\alpha}} \rho_{\mathbf{s}} \to 0.$$

is a split $\widetilde{T}$-exact sequence.

Proof: To see that the sequence splits, we find a $\widetilde{T}$-invariant complement for $\rho_{-\mathbf{s}+2}$ in $(\pi_{\mathbf{s}})_N$ that carries the representation $\rho_{\mathbf{s}}$. First, we find a complement on which $Z(\widetilde{T})$ acts by $\chi_{\mathbf{s}}$. Let $\{f_0, \cdots, f_{n-1}\}$ be a basis for $(\rho_{-\mathbf{s}+2}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2})$, identified with a subrepresentation of $(\pi_{\mathbf{s}})_N$, and let $\{g'_0, \cdots, g'_{n-1}\}$ be a set of coset representatives for a basis of $(\pi_{\mathbf{s}})_N/\rho_{-\mathbf{s}+2} \cong \rho_{\mathbf{s}}$. Therefore, for every $\mathtt{t} \in Z(\widetilde{T})$,

$$\mathtt{t} \cdot f_i = \chi_{-\mathbf{s}+2}(\mathtt{t}) f_i, \quad \text{and} \quad \mathtt{t} \cdot g'_i = \chi_{\mathbf{s}}(\mathtt{t}) g'_i + \sum_{j=0}^{n-1} \lambda_{ij}(\mathtt{t}) f_j,$$

for some $\lambda_{ij} : Z(\widetilde{T}) \to \mathbb{C}$. We will find constants c_{ij}, $0 \leq i, j < n$ such that $Z(\widetilde{T})$ acts on $g_i := g'_i + \sum_{j=0}^{n-1} c_{ij} f_j$ by the character $\chi_{\mathbf{s}}$, and hence $\{g_i \mid 0 \leq i < n\}$ is a basis for the $Z(\widetilde{T})$-complement to $\rho_{-\mathbf{s}+2}$. Observe that

$$\mathtt{t} \cdot g_i = \chi_{\mathbf{s}}(\mathtt{t}) g'_i + \sum_{j=0}^{n-1} \lambda_{ij}(\mathtt{t}) f_j + \sum_{j=0}^{n-1} c_{ij} \chi_{-\mathbf{s}+2}(\mathtt{t}) f_j,$$

for every $\mathtt{t} \in Z(\widetilde{T})$, which is equal to $\chi_{\mathbf{s}}(\mathtt{t}) g_i$ if and only if $\lambda_{ij}(\mathtt{t}) = c_{ij}(\chi_{\mathbf{s}}(\mathtt{t}) - \chi_{-\mathbf{s}+2}(\mathtt{t}))$. Let $\mathtt{t}_0 \in Z(\widetilde{T})$ be such that $\chi_{\mathbf{s}}(\mathtt{t}_0) - \chi_{-\mathbf{s}+2}(\mathtt{t}_0) \neq 0$; because $\chi_{\mathbf{s}}$ is regular, such a $\mathtt{t}_0$ exists. We claim that the relation $\mathtt{t} \cdot (\mathtt{t}_0 \cdot g_i) = \mathtt{t}_0 \cdot (\mathtt{t} \cdot g_i)$, for every $\mathtt{t} \in Z(\widetilde{T})$ and $0 \leq i, j < n$, implies

$$\lambda_{ij}(\mathtt{t}) \quad = \quad \frac{\lambda_{ij}(\mathtt{t}_0)(\chi_{\mathbf{s}}(\mathtt{t}) - \chi_{-\mathbf{s}+2}(\mathtt{t}))}{\chi_{\mathbf{s}}(\mathtt{t}_0) - \chi_{-\mathbf{s}+2}(\mathtt{t}_0)}.$$

Hence, by setting $c_{ij} = \frac{\lambda_{ij}(\mathtt{t}_0)}{\chi_{\mathbf{s}}(\mathtt{t}_0) - \chi_{-\mathbf{s}+2}(\mathtt{t}_0)}$, we have $\mathtt{t} \cdot g_i = \chi_{\mathbf{s}}(\mathtt{t}) g_i$, $0 \leq i < n$. Now, to prove the claim note that, for all $\mathtt{t} \in Z(\widetilde{T})$,

$$\mathtt{t} \cdot (\mathtt{t}_0 \cdot g_i) \quad = \quad \mathtt{t} \cdot (\chi_{\mathbf{s}}(\mathtt{t}_0) g_i + \sum_{j=1}^{m} \lambda_{ij}(\mathtt{t}_0) f_j)$$

$$= \chi_{\mathbf{s}}(\mathtt{t}_0)\big(\chi_{\mathbf{s}}(\mathtt{t})g_i + \sum_{j=1}^{m} \lambda_{ij}(\mathtt{t})f_j\big) + \sum_{j=1}^{m} \lambda_{ij}(\mathtt{t}_0)\chi_{-\mathbf{s}+2}(\mathtt{t})f_j.$$

Similarly $\mathtt{t}_0 \cdot (\mathtt{t} \cdot g_i) = \chi_{\mathbf{s}}(\mathtt{t})\big(\chi_{\mathbf{s}}(\mathtt{t}_0)g_i + \sum_{j=1}^{m} \lambda_{ij}(\mathtt{t}_0)f_j\big) + \sum_{j=1}^{m} \lambda_{ij}(\mathtt{t})\chi_{-\mathbf{s}+2}(\mathtt{t}_0)f_j$. Hence,

$$\chi_{\mathbf{s}}(\mathtt{t}_0)\sum_{j=1}^{n} \lambda_{ij}(\mathtt{t})f_j + \sum_{j=1}^{n} \lambda_{ij}(\mathtt{t}_0)\chi_{-\mathbf{s}+2}(\mathtt{t})f_j = \chi_{\mathbf{s}}(\mathtt{t})\sum_{j=1}^{n} \lambda_{ij}(\mathtt{t}_0)f_j + \sum_{j=1}^{n} \lambda_{ij}(\mathtt{t})\chi_{-\mathbf{s}+2}(\mathtt{t}_0)f_j.$$

So, $\lambda_{ij}(\mathtt{t})(\chi_{\mathbf{s}}(\mathtt{t}_0) - \chi_{-\mathbf{s}+2}(\mathtt{t}_0)) = \lambda_{ij}(\mathtt{t}_0)(\chi_{\mathbf{s}}(\mathtt{t}) - \chi_{-\mathbf{s}+2}(\mathtt{t}))$, for each i and j, and hence the claim. It remains to show that $W := \mathrm{Span}\{g_i \mid 0 \le i < n\}$ is a $\widetilde{T}$-invariant complement to $\rho_{-\mathbf{s}+2}$. Let $\mathtt{t}' \in \widetilde{T}$, $\mathtt{t} \in Z(\widetilde{T})$. Then, for $0 \le i < n$,

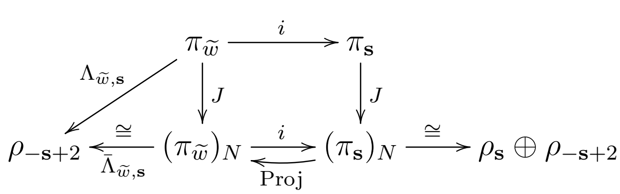
$$\mathtt{t} \cdot (\mathtt{t}' \cdot g_i) = \mathtt{t}' \cdot (\mathtt{t} \cdot g_i) = \mathtt{t}' \cdot (\chi_{\mathbf{s}}(\mathtt{t})g_i) = \chi_{\mathbf{s}}(\mathtt{t})(\mathtt{t}' \cdot g_i).$$

That is, $Z(\widetilde{T})$ acts on $\mathtt{t}'{\cdot}g_i$ by $\chi_{\mathbf{s}}$ which implies $\mathtt{t}'{\cdot}g_i \in W$, and that finishes the proof. ∎

Proposition 5.2.2. *Suppose $\chi_{\mathbf{s}}$ is a regular character. Then there exists a unique $\widetilde{B}$-equivariant morphism $\Lambda_{\mathbf{s}} : \pi_{\mathbf{s}}|_{\widetilde{B}} \to \rho_{-\mathbf{s}+2}$ that extends $\Lambda_{\widetilde{w},\mathbf{s}}$, and a $\widetilde{G}$-equivariant morphism $\mathcal{T}_{\mathbf{s}} : \pi_{\mathbf{s}} \to \pi_{-\mathbf{s}+2}$, such that for any $h \in \pi_{\mathbf{s}}$*

$$\mathcal{T}_{\mathbf{s}}(h)(1) = \Lambda_{\mathbf{s}}(h).$$

Proof: Consider the following extension of Figure 5.1. Observe that, by Lemma 5.2.1, the projection map from $\rho_{\mathbf{s}} \oplus \rho_{-\mathbf{s}+2}$ to $\rho_{-\mathbf{s}+2}$ defines a map Proj : $(\pi_{\mathbf{s}})_N \to \rho_{-\mathbf{s}+2}$. All the maps are $\widetilde{T}$-equivariant morphisms and trivial on N, so $\widetilde{B}$-equivariant morphisms.



Since the Jacquet functor is exact, this diagram commutes, in the sense that for $h \in \pi_{\widetilde{w}}$, $\Lambda_{\widetilde{w},\mathbf{s}}(h) = \bar{\Lambda}_{\widetilde{w},\mathbf{s}} \circ \mathrm{Proj} \circ J \circ i(h)$. We can extend $\Lambda_{\widetilde{w},\mathbf{s}}$ to a map $\Lambda_{\mathbf{s}} : \pi_{\mathbf{s}} \to \rho_{-\mathbf{s}+2}$

by defining $\Lambda_{\mathbf{s}}(h) = \bar{\Lambda}_{\widetilde{w},\mathbf{s}} \circ \mathrm{Proj} \circ J(h)$. Hence $\Lambda_{\mathbf{s}}$ is a $\widetilde{B}$-equivariant map. By Frobenius reciprocity, we have

$$\begin{aligned} \mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{-\mathbf{s}+2}) &\cong \mathrm{Hom}_{\widetilde{B}}(\pi_{\mathbf{s}}|_{\widetilde{B}}, \rho_{-\mathbf{s}+2}) \\ \phi &\mapsto \alpha \circ \phi, \end{aligned} \tag{5.2.1}$$

where α is the evaluation map at 1. Moreover, by Lemma 1.1.41

$$\mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{-\mathbf{s}+2}) \cong \mathrm{Hom}_{\widetilde{T}}((\pi_{\mathbf{s}})_N, \rho_{-\mathbf{s}+2}),$$

which because $(\pi_{\mathbf{s}})_N \cong \rho_{\mathbf{s}} \oplus \rho_{-\mathbf{s}+2}$, and $\rho_{\mathbf{s}} \ncong \rho_{\mathbf{s}}$, is one-dimensional. It follows that $\Lambda_{\mathbf{s}}$ is unique, up to a scalar. Let $\mathcal{T}_{\mathbf{s}} \in \mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{-\mathbf{s}+2})$ be the $\widetilde{G}$-map corresponding to $\Lambda_{\mathbf{s}} \in \mathrm{Hom}_{\widetilde{B}}(\pi_{\mathbf{s}}|_{\widetilde{B}}, \rho_{-\mathbf{s}+2})$. Because $\mathcal{T}_{\mathbf{s}}$ maps to $\Lambda_{\mathbf{s}}$, (5.2.1) implies that $\mathcal{T}_{\mathbf{s}}(f)(1) = \Lambda_{\mathbf{s}}(f)$. ∎

Remark 5.2.3. When $\mathrm{Re}(\mathbf{s}) > 1$, by Remark 5.1.10, the extension $\Lambda_{\mathbf{s}}$ of $\Lambda_{\widetilde{w},\mathbf{s}}$ is defined via the integral given in (5.1.7). Therefore, in this case, there is an explicit formula for the extension.

Similarly, we can construct a $\widetilde{G}$-map $\mathcal{T}_{-\mathbf{s}+2} : \pi_{-\mathbf{s}+2} \to \pi_{\mathbf{s}}$. So, the composition of these maps gives us an intertwining operator of $\pi_{\mathbf{s}}$, which we shall denote $\mathcal{T}$:

$$\mathcal{T} : \pi_{\mathbf{s}} \xrightarrow{\mathcal{T}_{\mathbf{s}}} \pi_{-\mathbf{s}+2} \xrightarrow{\mathcal{T}_{-\mathbf{s}+2}} \pi_{\mathbf{s}}. \tag{5.2.2}$$

Corollary 5.2.4. *Let* $\mathcal{T}$ *be as in* (5.2.2). *Then* $\mathcal{T} = \gamma(\mathbf{s})\mathrm{id}$*, for some constant* $\gamma(\mathbf{s}) \in \mathbb{C}$.

Proof: By Lemma 1.1.41, and Lemma 5.2.1

$$\mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{\mathbf{s}}) \cong \mathrm{Hom}_{\widetilde{T}}((\pi_{\mathbf{s}})_N, \rho_{\mathbf{s}}) \cong \mathrm{Hom}_{\widetilde{T}}(\rho_{-\mathbf{s}+2} \oplus \rho_{\mathbf{s}}, \rho_{\mathbf{s}}).$$

Because $\rho_{-\mathbf{s}+2}$ and $\rho_{\mathbf{s}}$ are distinct irreducible representations of $\widetilde{T}$, this intertwining space is one-dimensional. Hence, $\mathcal{T}$ is a scalar multiple of the identity. ∎

We will show that the vanishing of $\gamma(\mathbf{s})$ determines the reducibility points of $\pi_{\mathbf{s}}$.

5.2.1 Irreducibility Criteria

In the next proposition, we reduce the problem of the irreducibility of $\pi_{\mathbf{s}}$ to the one of calculating $\gamma(\mathbf{s})$, when $\chi_{\mathbf{s}}$ is a regular character. We will use the Projectivity theorem, which we state without proof. The proof, in a more general setting, can be found in [BH06, Appendix 10.a].

Theorem 5.2.5 (Projectivity Theorem)**.** *Let (π, V) be an irreducible genuine smooth representation of $\widetilde{G}$ such that $\pi_N = 0$, and let (τ, U) be a smooth genuine representation of $\widetilde{G}$. Let $f : U \to V$ be a surjective $\widetilde{G}$-homomorphism. There exists a $\widetilde{G}$-homomorphism $\phi : V \to U$ such that $f \circ \phi = id_V$.*

Proposition 5.2.6. *Assume $\chi_{\mathbf{s}}$ is regular. Let $\mathcal{T}$ be as in* (5.2.2)*. The principal series representation $\pi_{\mathbf{s}}$ is irreducible if and only if $\mathcal{T} \neq 0$.*

Proof: Suppose $\mathcal{T} = 0$. We have $\mathcal{T} = \mathcal{T}_{-\mathbf{s}+2} \circ \mathcal{T}_{\mathbf{s}}$. If $\mathcal{T}_{\mathbf{s}}$ is not injective, then the kernel of $\mathcal{T}_{\mathbf{s}}$ is a $\widetilde{G}$-invariant subspace of $\pi_{\mathbf{s}}$. Since $\mathcal{T}_{\mathbf{s}}$ is, by construction, not the zero map, its kernel is a proper subspace and hence, $\pi_{\mathbf{s}}$ is reducible.

Let us, for the purpose of contradiction, assume that $\mathcal{T}_{\mathbf{s}}$ is injective. Set $W :=$ Img$(\mathcal{T}_{\mathbf{s}})$. Observe that because $\mathcal{T}_{\mathbf{s}} \neq 0$, W is non-trivial. Moreover, because $\mathcal{T}_{-\mathbf{s}+2}(W) = 0$ and $\mathcal{T}_{-\mathbf{s}+2} \neq 0$, W is a proper subrepresentation of $(\pi_{-\mathbf{s}+2}, \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho_{-\mathbf{s}+2})$. Set $M := (\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}} \rho_{-\mathbf{s}+2})/W$. Because the Jacquet functor J is exact, the image of the exact sequence

$$0 \to \pi_{\mathbf{s}} \overset{\mathcal{T}_{\mathbf{s}}}{\hookrightarrow} \pi_{-\mathbf{s}+2} \overset{\mathsf{p}}{\twoheadrightarrow} M \to 0,$$

where p is the projection map, under J remains exact; that is

$$0 \to (\pi_{\mathbf{s}})_N \hookrightarrow (\pi_{-\mathbf{s}+2})_N \twoheadrightarrow M_N \to 0$$

is exact. Observe that $(\pi_{\mathbf{s}})_N \cong (\pi_{-\mathbf{s}+2})_N$, and they are finite-dimensional spaces, so $M_N = 0$. Let M' be an irreducible quotient of M, then $M'_N = 0$. By Projectivity Theorem 5.2.5, there exists a $\widetilde{G}$-homomorphism that embeds M' into $\pi_{-\mathbf{s}+2}$; hence

M', being a subrepresentation of a principal series representation $\pi_{-\mathbf{s}+2}$, has non-zero Jacquet module; that is $M'_N \neq 0$, which is a contradiction. Hence, $\mathcal{T}_{\mathbf{s}}$ is not injective, whence $\pi_{\mathbf{s}}$ is reducible.

Conversely, assume that $\mathcal{T} \neq 0$ and suppose, to the contrary, there exists a non-trivial subrepresentation of $\pi_{\mathbf{s}}$. Let W be a maximal non-trivial subrepresentation of $\pi_{\mathbf{s}}$, so that $M := \pi_{\mathbf{s}}/W$ is irreducible. The image of the exact sequence

$$0 \to W \overset{i}{\hookrightarrow} \pi_{\mathbf{s}} \overset{\mathsf{p}}{\twoheadrightarrow} M \to 0,$$

where i and p are inclusion and projection map respectively, remains exact under the Jacquet functor; that is

$$0 \to W_N \hookrightarrow (\pi_{\mathbf{s}})_N \twoheadrightarrow M_N \to 0 \tag{5.2.3}$$

is exact. The Jacquet module of W, being a subrepresentation of a principal series representation, is non-zero. Hence, W_N is equal to either $\rho_{\mathbf{s}}$, $\rho_{-\mathbf{s}+2}$, or $\rho_{\mathbf{s}} \oplus \rho_{-\mathbf{s}+2}$.

If $W_N = \rho_{\mathbf{s}} \oplus \rho_{-\mathbf{s}+2}$ then $M_N = 0$, which, similar to the argument above, is impossible by the Projectivity Theorem. So, W_N is isomorphic to either $\rho_{\mathbf{s}}$ or $\rho_{-\mathbf{s}+2}$, which, because (5.2.3) is exact, implies that M_N is isomorphic to either $\rho_{-\mathbf{s}+2}$ or $\rho_{\mathbf{s}}$.

If $M_N \cong \rho_{\mathbf{s}}$ by Lemma 1.1.41

$$\mathrm{Hom}_{\widetilde{T}}(M_N, \rho_{\mathbf{s}}) = \mathrm{Hom}_{\widetilde{G}}(M, \pi_{\mathbf{s}}),$$

which implies there exists a non-zero $\widetilde{G}$-map $\mathsf{q} : M \to \pi_{\mathbf{s}}$. Hence, $\mathsf{q} \circ \mathsf{p} : \pi_{\mathbf{s}} \overset{\mathsf{p}}{\to} M \overset{\mathsf{q}}{\to} \pi_{\mathbf{s}}$ is a $\widetilde{G}$-intertwining operator. Note that $\mathsf{q} \circ \mathsf{p}$ is not injective, because p is not injective. This contradicts the fact that $\mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{\mathbf{s}}) \cong \mathbb{C}$.

Similarly, suppose $M_N \cong \rho_{-\mathbf{s}+2}$. Note that $\mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{-\mathbf{s}+2}) = \mathbb{C}\mathcal{T}_{\mathbf{s}}$. Moreover, the assumption $\mathcal{T} = \mathcal{T}_{-\mathbf{s}+2} \circ \mathcal{T}_{\mathbf{s}} \neq 0$ implies $\mathcal{T}_{\mathbf{s}}$ is injective. By Lemma 1.1.41

$$\mathrm{Hom}_{\widetilde{T}}(M_N, \rho_{-\mathbf{s}+2}) = \mathrm{Hom}_{\widetilde{G}}(M, \pi_{-\mathbf{s}+2}).$$

So, there exists a non-zero $\widetilde{G}$-map $\mathsf{q} : M \to \pi_{-\mathbf{s}+2}$. Hence, the map $\mathsf{q} \circ \mathsf{p} : \pi_{\mathbf{s}} \overset{\mathsf{p}}{\to} M \overset{\mathsf{q}}{\to} \pi_{-\mathbf{s}+2}$ is a non-injective map in $\mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{-\mathbf{s}+2})$, which is a contradiction. Hence, such a W does not exist, and $\pi_{\mathbf{s}}$ is irreducible. ∎

By Proposition 5.2.6, to determine the reducibility points of $\pi_{\mathbf{s}}$, we need to know all the values of $\mathbf{s}$ such that $\gamma(\mathbf{s}) = 0$. In order to find $\gamma(\mathbf{s})$, it suffices to determine the image of the spherical vector $\phi_{\mathbf{s}}$ under $\mathcal{T} = \mathcal{T}_{-\mathbf{s}+2} \circ \mathcal{T}_{\mathbf{s}}$.

Let $f_0' \in \mathrm{Ind}_A^{\widetilde{T}} \chi_{-\mathbf{s}+2}$ be such that $\mathrm{Supp} f_0' \subset A$ and $f_0'(1) = 1$, and let $\phi_{-\mathbf{s}+2} : \widetilde{G} \to \mathrm{Ind}_A^{\widetilde{T}} \chi_{-\mathbf{s}+2}$, defined via $\phi_{-\mathbf{s}+2}(\mathtt{g}) = \rho_{-\mathbf{s}+2}(\mathtt{t}) f_0'$, whenever $\mathtt{g} = \mathtt{ntk}$, $\mathtt{n} \in N, \mathtt{t} \in \widetilde{T}$ and $\mathtt{k} \in \widetilde{K}_0$, be the normalized spherical function in $\pi_{-\mathbf{s}+2}$.

Lemma 5.2.7. *Let $\mathcal{T}_{\mathbf{s}}$ be as in Proposition 5.2.2, and let $\phi_{\mathbf{s}}$ and $\phi_{-\mathbf{s}+2}$ be the normalized spherical functions in ${\pi_{\mathbf{s}}}^{\widetilde{K}_0}$ and ${\pi_{-\mathbf{s}+2}}^{\widetilde{K}_0}$ respectively. Then there exists a scalar $c(\mathbf{s}) \in \mathbb{C}$ such that*

$$\mathcal{T}_{\mathbf{s}}(\phi_{\mathbf{s}}) = c(\mathbf{s}) \phi_{-\mathbf{s}+2}.$$

Proof: Since $\mathcal{T}_{\mathbf{s}}$ is an intertwining operator, it takes the $\widetilde{K}_0$-fixed subspace of $\pi_{\mathbf{s}}$ to the $\widetilde{K}_0$-fixed subspace of $\pi_{-\mathbf{s}+2}$. By Lemma 5.1.3, $\pi_{\mathbf{s}}^{\widetilde{K}_0}$ and $\pi_{-\mathbf{s}+2}^{\widetilde{K}_0}$ are one-dimensional spaces, generated by $\phi_{\mathbf{s}}$ and $\phi_{-\mathbf{s}+2}$ respectively. Hence, the lemma follows. ∎

Similarly, $\mathcal{T}_{-\mathbf{s}+2}(\phi_{-\mathbf{s}+2}) = c(-\mathbf{s}+2)\phi_{\mathbf{s}}$, for some constant $c(-\mathbf{s}+2)$. Since $\gamma(\mathbf{s}) = c(\mathbf{s})c(-\mathbf{s}+2)$, our goal is to calculate the constants $c(\mathbf{s})$ and $c(-\mathbf{s}+2)$.

Lemma 5.2.8. *Let $c(\mathbf{s}) \in \mathbb{C}$ be such that $\mathcal{T}_{\mathbf{s}}(\phi_{\mathbf{s}}) = c(\mathbf{s})\phi_{-\mathbf{s}+2}$, and let $\Lambda_{\mathbf{s}} : \pi_{\mathbf{s}} \to \rho_{-\mathbf{s}+2}$ be as in Proposition 5.2.2. Then $c(\mathbf{s}) f_0' = \Lambda_{\mathbf{s}}(\phi_{\mathbf{s}})$.*

Proof: By Proposition 5.2.2, $\Lambda_{\mathbf{s}}(\phi_{\mathbf{s}}) = \mathcal{T}_{\mathbf{s}}(\phi_{\mathbf{s}})(1)$, which by Lemma 5.2.7 equals $c(\mathbf{s})\phi_{-\mathbf{s}+2}(1)$. Since $\phi_{-\mathbf{s}+2}(1) = f_0'$ by definition, the result follows. ∎

Therefore, in order to calculate $c(\mathbf{s})$, it is enough to calculate $\Lambda_{\mathbf{s}}(\phi_{\mathbf{s}})$. However, we have no explicit formula for $\Lambda_{\mathbf{s}}$, unless $\mathrm{Re}(\mathbf{s}) > 1$. Let φ_0 be the quasi-characteristic function defined in (5.1.4). Recall from Lemma 5.1.8 that $\phi_{\mathbf{s}} = P_{\mathbf{s}}(\varphi_0)$, where $P_{\mathbf{s}} : C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}} \chi_{\mathbf{s}}) \to \pi_{\mathbf{s}}$ is given by

$$P_{\mathbf{s}}(\varphi)(\mathtt{g}) = \int_{\widetilde{B}} \rho_{\mathbf{s}}(\mathtt{b}^{-1}) \varphi(\mathtt{bg}) d\mathtt{b}.$$

Hence, $\Lambda_{\mathbf{s}}(\phi_{\mathbf{s}}) = \Lambda_{\mathbf{s}} \circ P_{\mathbf{s}}(\varphi_0)$. Denote the map $\Lambda_{\mathbf{s}} \circ P_{\mathbf{s}}$ by $D_{\mathbf{s}}$:

$$\begin{aligned} D_{\mathbf{s}} : C_c^{\infty}(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}) &\rightarrow \rho_{-\mathbf{s}+2} \\ h &\rightarrow \Lambda_{\mathbf{s}} \circ P_{\mathbf{s}}(h). \end{aligned} \tag{5.2.4}$$

Recall from Proposition 5.2.2 that, under the isomorphism in Lemma 5.1.12, $\Lambda_{\mathbf{s}}|_{\mathrm{In}_{\widetilde{w}}}$ is given by

$$\begin{aligned} \Lambda_{\widetilde{w},\mathbf{s}} : \mathrm{In}_{\widetilde{w}} &\rightarrow \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2} \\ h &\mapsto \int_N h(\widetilde{w}\mathtt{n})d\,\mathtt{n}. \end{aligned}$$

Let $C_c^{\infty}(\widetilde{B}\widetilde{w}\widetilde{B}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ denote the subspace of $C_c^{\infty}(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$ that consists of those functions with support contained in $\widetilde{B}\widetilde{w}\widetilde{B}$. It is not difficult to see that if $\mathrm{Supp}(\varphi) \subseteq \widetilde{B}\widetilde{w}\widetilde{B}$ then $\mathrm{Supp}\big(P_{\mathbf{s}}(\varphi)\big) \subseteq \widetilde{B}\widetilde{w}\widetilde{B}$ as well, so $\mathrm{Img}(P_{\mathbf{s}}|_{C_c^{\infty}(\widetilde{B}\widetilde{w}\widetilde{B}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})}) \subseteq \mathrm{In}_{\widetilde{w}}$. Hence,

$$D_{\mathbf{s}}|_{C_c^{\infty}(\widetilde{B}\widetilde{w}\widetilde{B}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})} = \Lambda_{\widetilde{w},\mathbf{s}} \circ P_{\mathbf{s}},$$

for which we are going to give a formula, in the next lemma. First, observe that the map

$$\begin{aligned} \widetilde{B} \times N &\rightarrow \widetilde{B}\widetilde{w}N \\ (\mathtt{b}, \mathtt{n}) &\mapsto \mathtt{b}\widetilde{w}\mathtt{n} \end{aligned} \tag{5.2.5}$$

is a bijection.

Lemma 5.2.9. *Let $h \in C_c^{\infty}(\widetilde{B}\widetilde{w}\widetilde{B}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$, and let $D_{\mathbf{s}} = \Lambda_{\widetilde{w},\mathbf{s}} \circ P_{\mathbf{s}}$. Then, after a suitable normalization of the restriction of the Haar measure of $\widetilde{G}$ to $\widetilde{B}\widetilde{w}\widetilde{B}$,*

$$D_{\mathbf{s}}(h) = \int_{\widetilde{B}w\widetilde{B}} \psi(\mathtt{x})h(\mathtt{x})d\mathtt{x},$$

where $\psi(\mathtt{x}) = \delta_{\widetilde{B}}(\mathtt{b})\rho_{\mathbf{s}}(\mathtt{b}^{-1})$ for $\mathtt{x} = \mathtt{b}\widetilde{w}\mathtt{n}'$ with $\mathtt{n}' \in N$, $\mathtt{b} \in \widetilde{B}$.

Proof: Consider the open subset $\widetilde{B}\widetilde{w}\widetilde{B} = \widetilde{B}\widetilde{w}N$ of $\widetilde{G}$ with the measure μ inherited from the Haar measure of $\widetilde{G}$. Note that this measure is invariant under multiplication by N on the right and multiplication by $\widetilde{B}$ on the left. By (5.2.5), $\widetilde{B}\widetilde{w}N$ is in bijection with $\widetilde{B}\times N$. The uniqueness of Haar measure up to a scalar implies that any N-right invariant and $\widetilde{B}$-left invariant Haar measure on $\widetilde{B}\times N$ is a scalar multiple of μ.

Now, let $f \in C_c^\infty(\widetilde{B}\widetilde{w}\widetilde{B}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$. Recall that we assume that the Haar measure on $\widetilde{B}$ is right invariant. For clarity, we subscribe the right and left invariant integrals by R and L consecutively. So,

$$\begin{aligned} D_{\mathbf{s}}(h) &= \Lambda_{\widetilde{w},\mathbf{s}}(P_{\mathbf{s}}(h)) \\ &= \int_N \int_{\widetilde{B}} \rho_{\mathbf{s}}(\mathtt{b}^{-1}) h(\mathtt{b}\widetilde{w}\mathtt{n}) d_R\mathtt{b} d_R\mathtt{n} \\ &= \int_N \int_{\widetilde{B}} \delta_{\tilde{B}}(\mathtt{b})\rho_{\mathbf{s}}(\mathtt{b}^{-1}) h(\mathtt{b}\widetilde{w}\mathtt{n}) d_L\mathtt{b} d_R\mathtt{n}. \end{aligned} \tag{5.2.6}$$

Let $\psi(\mathtt{x}) = \delta_{\tilde{B}}(\mathtt{b})\rho_{\mathbf{s}}(\mathtt{b}^{-1})$ for $\mathtt{x} = \mathtt{b}\widetilde{w}\mathtt{n}'$, $\mathtt{n}' \in N$, $\mathtt{b} \in \widetilde{B}$. Then (5.2.6) can be written as

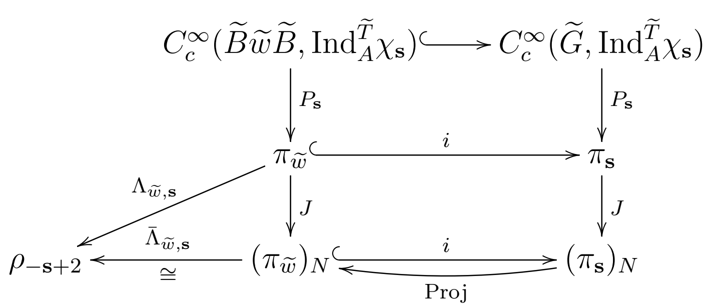
$$\int_{\widetilde{B}\times N} \psi(\mathtt{x}) h(\mathtt{x}) d\mathtt{x}.$$

This is an N-right invariant and $\widetilde{B}$-left invariant integral on the set $\widetilde{B}\times N$. Therefore it is equal to

$$c\int_{\widetilde{B}\widetilde{w}\widetilde{B}} \psi(\mathtt{x}) h(\mathtt{x}) d\mathtt{x},$$

for some non-zero constant c. We can normalize the Haar measure on $\widetilde{B}\widetilde{w}\widetilde{B}$ so that the constant $c = 1$. ∎

Consider the following extension of Figure 5.1.



This diagram commutes. Note that $D_{\mathbf{s}} : C_c^\infty(\widetilde{G}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}) \to \rho_{-\mathbf{s}+2}$, which is given by $\bar{\Lambda}_{\widetilde{w},\mathbf{s}} \circ \mathrm{Proj} \circ J \circ P_{\mathbf{s}}$, is the extension of $\Lambda_{\widetilde{w},\mathbf{s}} \circ P_{\mathbf{s}}$, for which Lemma 5.2.9 gives the formula. By Remark 5.2.3, when $\mathrm{Re}(\mathbf{s}) > 1$, $D_{\mathbf{s}}$ is also given by the formula in Lemma 5.2.9. Next, we state the following proposition without proof.

Proposition 5.2.10. *For every $h \in C_c^\infty(\widetilde{G}, \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}})$, the function $D_{\mathbf{s}}(h)$ is a holomorphic function of $\mathbf{s} \in \mathbb{C}$.*

The main idea of the proof is to construct a function $\widetilde{h}$, for every function $h \in C_c^\infty(\widetilde{G}, \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}})$, such that $\mathrm{Supp}(\widetilde{h}) \subseteq \widetilde{B}\widetilde{w}\widetilde{B}$ and $\Lambda_{\mathbf{s}}(h) = \Lambda_{\mathbf{s}}(\widetilde{h})$. Because $\Lambda_{\mathbf{s}}(\widetilde{h})$ is given by an integral, which by Lemma 5.1.9, has a compactly supported integrand, it depends holomorphically on $\mathbf{s}$, and hence the result follows. A similar statement is proved in [CS80, Proposition 2.1].

In the next proposition, using the formula for $D_{\mathbf{s}}$, we calculate $\mathcal{T}_{\mathbf{s}}(\phi_{\mathbf{s}})$. Let us prove the following technical lemma first.

Lemma 5.2.11. *Let $\Omega_m = \{\left(\left(\begin{smallmatrix} a & b \\ \varpi^m c & d \end{smallmatrix}\right), \zeta\right) \in \widetilde{I} |\, a, b, c, d \in \mathcal{O}\}$. Then Ω_m is a group and $[\widetilde{K} : \Omega_m] = q^m + q^{m-1}$.*

Proof: It is not difficult to see that Ω_m is a group. Let P be the canonical projection from $\widetilde{K}$ onto $\widetilde{\mathrm{SL}}_2(\mathcal{O}/\mathfrak{p}^m)$. Define the set $\overline{\Omega_m} = \{\left(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right), \zeta\right) | a, b \in \mathcal{O}/\mathfrak{p}^m\}$. Then $\Omega_m = P^{-1}(\overline{\Omega_m})$ and hence $[\widetilde{K} : \Omega_m] = [\widetilde{\mathrm{SL}}_2(\mathcal{O}/\mathfrak{p}^m) : \overline{\Omega_m}]$. Let us calculate the cardinality of $\widetilde{\mathrm{SL}}_2(\mathcal{O}/\mathfrak{p}^m) = \{\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta\right) |\, a, b, c, d \in \mathcal{O}/\mathfrak{p}^m, ad - bc = 1\}$. Note that the determinant condition $ad - bc = 1$ in $\mathcal{O}/\mathfrak{p}^m$ implies that a and b cannot both lie in $\mathfrak{p}/\mathfrak{p}^m$. Hence, we can write $\mathrm{SL}_2(\mathcal{O}/\mathfrak{p}^m)$ as disjoint union $A \cup B \cup C$ where

1. $A = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}/\mathfrak{p}^m) \mid a \in \mathfrak{p}/\mathfrak{p}^m, b \notin \mathfrak{p}/\mathfrak{p}^m, ad - bc = 1\}$,

2. $B = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}/\mathfrak{p}^m) \mid b \in \mathfrak{p}/\mathfrak{p}^m, a \notin \mathfrak{p}/\mathfrak{p}^m, ad - bc = 1\}$,

3. $C = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}/\mathfrak{p}^m) \mid a, b \notin \mathfrak{p}/\mathfrak{p}^m, ad - bc = 1\}$.

Elementary counting shows that $|A| = |B| = (q^m - q^{m-1})q^{m-1}q^m$ and $|C| = (q^m - q^{m-1})^2 q^m$ elements. So, $|\widetilde{\mathrm{SL}}_2(\mathcal{O}/\mathfrak{p}^m)| = q^m(q^m - q^{m-1})(q^m + q^{m-1})n$. One can see that $|\overline{\Omega_m}| = (q^m - q^{m-1})q^m n$ and therefore, $[\widetilde{\mathrm{SL}}_2(\mathcal{O}/\mathfrak{p}^m) : \overline{\Omega_m}] = q^m + q^{m-1}$. ∎

Proposition 5.2.12. *Let $\mathcal{T}_{\mathbf{s}}$ be as in Proposition 5.2.2 and $\phi_{\mathbf{s}}$ and $\phi_{-\mathbf{s}+2}$ be the normalized spherical functions in ${\pi_{\mathbf{s}}}^{\widetilde{K}_0}$ and ${\pi_{-\mathbf{s}+2}}^{\widetilde{K}_0}$ respectively. Then*

$$\mathcal{T}_{\mathbf{s}}(\phi_{\mathbf{s}}) = c(\mathbf{s})\phi_{-\mathbf{s}+2},$$

where

$$c(\mathbf{s}) = \frac{1}{q+1}\Big(\frac{1 - q^{-n\mathbf{s}+n-1}}{1 - q^{-n\mathbf{s}+n}}\Big).$$

Proof: First assume $\mathrm{Re}(\mathbf{s}) > 1$. By Lemma 5.2.8, $c(\mathbf{s})f_0' = \Lambda_{\mathbf{s}}(\phi_{\mathbf{s}})$, which, by (5.2.4), is equal to $D_{\mathbf{s}}(\varphi_0)$. By Lemma 5.2.9

$$D_{\mathbf{s}}(h) = \int_{\widetilde{B}w\widetilde{B}} \psi(\mathtt{x})h(\mathtt{x})d\mathtt{x}, \tag{5.2.7}$$

when $h \in C_c^{\infty}(\widetilde{B}\widetilde{w}\widetilde{B}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}})$. We apply (5.2.7) to φ_0. Note that since φ_0 is 0 outside of $\widetilde{K}$, one can restrict the domain of the integral in (5.2.7) to $\widetilde{B}\widetilde{w}\widetilde{B} \cap \widetilde{K}$. The decomposition $\widetilde{K} = \widetilde{I} \cup \widetilde{I}\widetilde{w}\widetilde{I}$ yields $D_{\mathbf{s}}(\varphi_0)$ is equal to

$$\int_{(\widetilde{B}w\widetilde{B})\cap(\widetilde{I}w\widetilde{I})} \psi(\mathtt{x})\varphi_0(\mathtt{x})d\mathtt{x} + \int_{(\widetilde{B}w\widetilde{B})\cap(\widetilde{I})} \psi(\mathtt{x})\varphi_0(\mathtt{x})d\mathtt{x}. \tag{5.2.8}$$

Since for every $\mathtt{x} = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta)$ in $(\widetilde{B}\widetilde{w}\widetilde{B}) \cap \widetilde{K}$, we have $c \neq 0$, it follows that

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \zeta\right) = \underbrace{(\mathrm{ut}(ac^{-1}), 1)(\mathrm{dg}(-c^{-1}), \zeta)}_{\mathtt{b}}\ \widetilde{w}\ \underbrace{(\mathrm{ut}(dc^{-1}), 1)}_{\mathtt{n}'}. \tag{5.2.9}$$

Recall from Lemma 5.2.9, $\psi(\mathtt{x}) = \delta_{\widetilde{B}}(\mathtt{b})\rho_{\mathbf{s}}(\mathtt{b}^{-1})$ for $\mathtt{x} = \mathtt{b}\widetilde{w}\mathtt{n}'$, $\mathtt{n}' \in N$, $\mathtt{b} \in \widetilde{B}$.

We calculate the first integral in (5.2.8). Assume $\mathtt{x} = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta) \in (\widetilde{B}w\widetilde{B}) \cap (\widetilde{I}w\widetilde{I})$, so that $c \in \mathcal{O}^{\times}$. Therefore, $\mathtt{x} = (1, \zeta)\,(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), 1)$, where $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), 1) \in \widetilde{K}_0$. It follows from the definition of φ_0 in (5.1.4) that in this case

$$\psi(\mathtt{x})\varphi_0(\mathtt{x}) = \delta_{\widetilde{B}}(\mathrm{dg}(-c^{-1}), \zeta)\rho_{\mathbf{s}}\big(\,(\mathrm{dg}(-c), \zeta^{-1})\,\big)\epsilon(\zeta)M^{-1}f_0.$$

Observe that $\rho_{\mathbf{s}}\big(\,(\mathrm{dg}(-c),\zeta^{-1})\,\big)=\rho_{\mathbf{s}}\,(\iota(-c))\,\rho_{\mathbf{s}}\,(1,\zeta^{-1})=\epsilon(\zeta^{-1})\rho_{\mathbf{s}}\,(\iota(-c))$; also, because $c\in\mathcal{O}^\times$, $\delta_{\tilde{B}}(\mathrm{dg}(-c^{-1}),\zeta)=1$, and by Lemma 5.1.5, $\rho_{\mathbf{s}}\,(\iota(-c))\,f_0=f_0$. Hence, $\psi(\mathtt{x})\varphi_0(\mathtt{x})=M^{-1}f_0$. It follows from Lemma 3.3.8 that $\widetilde{I}\widetilde{w}\widetilde{I}=(\widetilde{B}\cap\widetilde{K})\widetilde{w}(N\cap\widetilde{K})\subseteq\widetilde{B}\widetilde{w}\widetilde{B}$. Therefore, Lemma 3.3.9 implies that

$$\int_{(\widetilde{B}w\widetilde{B})\cap(\widetilde{I}w\widetilde{I})}\psi(\mathtt{x})\varphi_0(\mathtt{x})d\mathtt{x}=\mu\big((\widetilde{B}\cap\widetilde{K})\widetilde{w}(N\cap\widetilde{K})\big)M^{-1}f_0=\frac{q}{q+1}M^{-1}f_0. \quad (5.2.10)$$

Next we calculate the second integral in (5.2.8). For $m\geq 1$, define the open set

$$\widetilde{I}_m=\left\{(\begin{pmatrix} a & b\\ \varpi^m c & d\end{pmatrix},\zeta)\mid a,b,d\in\mathcal{O},c\in\mathcal{O}^\times\right\}.$$

Note that $(\widetilde{B}\widetilde{w}\widetilde{B})\cap\widetilde{I}=\bigcup_{m=1}^{\infty}\widetilde{I}_m$. Let $\mathtt{x}\in\widetilde{I}_m$, then $\mathtt{x}=\big(\left(\begin{smallmatrix} a & b\\ \varpi^m c & d\end{smallmatrix}\right),\zeta\big)$ for $c\in\mathcal{O}^\times$ and $a,b,d\in\mathcal{O}$. The determinant condition $ad-bc\varpi^m=1$ implies that $a,d\in\mathcal{O}^\times$. Applying the decomposition (5.2.9) to $\mathtt{x}$ shows that $\psi(\mathtt{x})$ is equal to

$$\delta_{\tilde{B}}\big(\,(\mathrm{dg}(-\varpi^{-m}c^{-1}),\zeta)\,\big)\rho_{\mathbf{s}}\big(\,(\mathrm{dg}(-\varpi^{-m}c^{-1}),\zeta)^{-1}\,\big).$$

Observe that $(\mathrm{dg}(-\varpi^{-m}c^{-1}),\zeta)^{-1}=(\mathrm{dg}(-\varpi^m c),\zeta^{-1}(-\varpi^m c,-\varpi^{-m}c^{-1})_n)$. It is not difficult to see that $(-\varpi^m c,-\varpi^{-m}c^{-1})_n=(-\varpi^m,-\varpi^{-m})_n=(\varpi^m,\varpi^m)_n=1$, because n is odd. Therefore, $\psi(\mathtt{x})$ is equal to

$$|-\varpi^{-m}c^{-1}|^2\rho_{\mathbf{s}}\left(\mathrm{dg}(-\varpi^m c),\zeta^{-1}\right)=q^{2m}\rho_{\mathbf{s}}\left(\mathrm{dg}(-\varpi^m c),\zeta^{-1}\right).$$

Note that $(\mathrm{dg}(-\varpi^m c),\zeta^{-1})=(\mathrm{I}_2,\zeta^{-1}(\varpi^m,-c)_n)\,\iota(\varpi^m)\iota(-c)$, and that

$$\rho_{\mathbf{s}}\left(\mathrm{I}_2,\zeta^{-1}(\varpi^m,-c)_n\right)=\epsilon(\zeta^{-1})\epsilon\left((\varpi^m,-c)_n\right).$$

Hence, $\psi(\mathtt{x})=q^{2m}\epsilon\big((\zeta^{-1})\big)\epsilon\big((\varpi^m,-c)_n\big)\rho_{\mathbf{s}}\,(\iota(\varpi^m))\,\rho_{\mathbf{s}}\,(\iota(-c))$.

To calculate $\varphi_0(\mathtt{x})$, observe that $\mathtt{x}=\big(\left(\begin{smallmatrix} a & b\\ \varpi^m c & d\end{smallmatrix}\right),\zeta\big)$ decomposes as

$$\left(\begin{pmatrix} a & b\\ \varpi^m c & d\end{pmatrix},(\varpi^m c,d)_n^{-1}\right)(\mathrm{I}_2,(\varpi^m c,d)_n\zeta),$$

where the first term is in $\widetilde{K}_0$ and the second is in the central μ_n. So, $\varphi_0(\mathtt{x}) = M^{-1}\epsilon\big((\varpi^m c, d)_n\big)\epsilon\big(\zeta\big) f_0$.

Consequently,

$$\psi(\mathtt{x})\varphi_0(\mathtt{x}) = M^{-1}q^{2m}\epsilon\big((\varpi^m c, d)_n\big)\epsilon\big(\zeta\big)\epsilon\big(\zeta^{-1}\big)\epsilon\big((\varpi^m, -c)_n\big)\rho_{\mathbf{s}}\left(\iota(\varpi^m)\right)\rho_{\mathbf{s}}\big(\left(\iota(-c)\right) f_0.$$

By Lemma 5.1.5, $\iota(-c)f_0 = f_0$. Set $f_m = \rho\left(\iota(\varpi^m)\right) f_0$. Moreover, because $c, d \in \mathcal{O}^\times$, $(\varpi^m c, d)_n(\varpi^m, -c)_n = (\varpi^m, -cd)_n(c, d)_n = (\varpi^m, -cd)_n = \vartheta^m_{\mathcal{O}^\times}(-cd)$. Hence,

$$\psi(\mathtt{x})\varphi_0(\mathtt{x}) = M^{-1}q^{2m}\epsilon\big(\vartheta^m_{\mathcal{O}^\times}(-cd)\big) f_m.$$

Recall that $\vartheta^m_{\mathcal{O}^\times} = 1$ if and only if $n|m$. For $\mathtt{x} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \zeta\right) \in \widetilde{I}_m$ define $g(\mathtt{x}) = \epsilon\left(\vartheta^m_{\mathcal{O}^\times}(-cd)\right)$. If $n \nmid m$, then $g(\mathtt{x}) : \widetilde{I}_m \to \mathbb{C}$ is a non-trivial character of $\widetilde{I}_m$ and hence, $\int_{\widetilde{I}_m} g(\mathtt{x})d\mathtt{x} = 0$. Therefore,

$$\int_{(\widetilde{B}\widetilde{w}\widetilde{B})\cap(\widetilde{I})} \psi(\mathtt{x})\varphi_0(\mathtt{x})d\mathtt{x} = \sum_{m=1}^{\infty} \int_{\widetilde{I}_m} M^{-1}q^{2m} g(\mathtt{x}) f_m d\mathtt{x} = M^{-1}\sum_{j=1}^{\infty} q^{2nj}\mu(\widetilde{I}_{nj}) f_{nj}. \tag{5.2.11}$$

Observe that if $n|m$, then $\iota(\varpi^m) \in Z(\widetilde{T})$, and thus

$$f_m = \rho_{\mathbf{s}}\left(\iota(\varpi^m)\right) f_0 = \chi_{\mathbf{s}}\left(\iota(\varpi^m)\right) f_0 = |\varpi^m|^{\mathbf{s}} f_0 = q^{-m\mathbf{s}} f_0.$$

Moreover, observe that $\widetilde{I}_m = \Omega_m - \Omega_{m+1}$, where $\Omega_m = \{\big(\left(\begin{smallmatrix} a & b \\ \varpi^m c & d \end{smallmatrix}\right), \zeta\big) \in \widetilde{I} |\, a, b, c, d \in \mathcal{O}\}$. Hence, $\mu(\widetilde{I}_m) = \mu(\Omega_m - \Omega_{m+1})$, which by Lemma 5.2.11 equals

$$\frac{1}{q^m + q^{m-1}} - \frac{1}{q^{m+1} + q^m} = \frac{q-1}{q^m(q+1)}.$$

Hence,

$$\int_{(\widetilde{B}\widetilde{w}\widetilde{B})\cap(\widetilde{I})} \psi(\mathtt{x})\varphi_0(\mathtt{x})d\mathtt{x} = M^{-1}\frac{q-1}{q+1}\sum_{j=1}^{\infty} q^{nj}q^{-nj\mathbf{s}} f_0 = M^{-1}\frac{q-1}{q+1}\sum_{j=1}^{\infty} q^{nj(1-\mathbf{s})} f_0. \tag{5.2.12}$$

Note that since $\mathrm{Re}(\mathbf{s}) > 1$, we have

$$\sum_{j=1}^{\infty} q^{j(-n\mathbf{s}+n)} = \frac{1}{1-q^{-n\mathbf{s}+n}} - 1 = \frac{1}{q^{n\mathbf{s}-n}-1}.$$

Hence, (5.2.12) is equal to

$$M^{-1}\frac{1}{q+1}\left(\frac{q-1}{(q^{n\mathbf{s}-n}-1)}\right) f_0. \tag{5.2.13}$$

Therefore, it follows from (5.2.10) and (5.2.13) that

$$\begin{aligned} D(\varphi_0) &= M^{-1}\left[\frac{q}{q+1}f_0 + \frac{1}{q+1}\left(\frac{q-1}{(q^{n\mathbf{s}-n}-1)}\right) f_0\right] \\ &= M^{-1}\left[\frac{1}{q+1}\left(\frac{1-q^{-n\mathbf{s}+n-1}}{1-q^{-n\mathbf{s}+n}}\right) f_0\right]. \end{aligned}$$

Hence, $c(\mathbf{s}) = \mu(\widetilde{B}\cap\widetilde{K})\frac{1}{q+1}\left(\frac{1-q^{-n\mathbf{s}+n-1}}{1-q^{-n\mathbf{s}+n}}\right)$, whenever $\mathrm{Re}(\mathbf{s}) > 1$. Since $D_{\mathbf{s}}$ is a holomorphic function in $\mathbf{s}$, it follows that the formula for $c(\mathbf{s})$ is valid for all $\mathbf{s}\in\mathbb{C}$, whenever it is defined. ∎

Evidently, we also have $c(-\mathbf{s}+2) = \mu(\widetilde{B}\cap\widetilde{K})\frac{1}{q+1}\left(\frac{1-q^{n\mathbf{s}-n-1}}{1-q^{n\mathbf{s}-n}}\right)$. Now we can give a reducibility criterion for $\pi_{\mathbf{s}}$ with a regular central character.

Theorem 5.2.13. *For a regular character $\chi_{\mathbf{s}}$, $\pi_{\mathbf{s}}$ is reducible if and only if $\mathbf{s} = 1 \pm \frac{1}{n}$.*

Proof: Note that $\mathcal{T}(\phi_{\mathbf{s}}) = \mathcal{T}_{-\mathbf{s}+2}(\mathcal{T}_{\mathbf{s}}(\phi_{\mathbf{s}})) = c(\mathbf{s})\mathcal{T}_{-\mathbf{s}+2}(\phi_{-\mathbf{s}+2}) = c(\mathbf{s})c(-\mathbf{s}+2)\phi_{\mathbf{s}}$. Hence, $\mathcal{T} = \gamma(\mathbf{s})\mathrm{id} = c(\mathbf{s})c(-\mathbf{s}+2)\mathrm{id}$. By Proposition 5.2.6, $\pi_{\mathbf{s}}$ is reducible if and only if $c(\mathbf{s})c(-\mathbf{s}+2) = 0$. If $c(\mathbf{s}) = 0$, by Proposition 5.2.12, $1 - q^{-n\mathbf{s}+n-1} = 0$ which implies $\mathbf{s} = 1 - \frac{1}{n} - \frac{2\pi i k}{n\log q}$, $k\in\mathbb{Z}$. If $c(-\mathbf{s}+2) = 0$, then, by Proposition 5.2.12, $1 - q^{n\mathbf{s}-n-1} = 0$, which implies $\mathbf{s} = 1 + \frac{1}{n} + \frac{2\pi i k}{n\log q}$, $k\in\mathbb{Z}$. The result follows from noting that $\chi_{\mathbf{s}} = \chi_{\mathbf{s}+\frac{2\pi i}{n\log q}}$, $k\in\mathbb{Z}$. ∎

5.3 The Case of Non-Regular Characters

Assume that $\chi_{\mathbf{s}}$ is not regular. Therefore, $\mathbf{s} = 1$ or $\mathbf{s} = 1 + \frac{\pi i}{\log q}$. In this case, our argument in Section 5.2 fails, because the sequence in Lemma 5.2.1 fails to split. On the other hand, observe that, in this case, the principal series representation $\pi_{\mathbf{s}}$ is unitary.

We will use the following subgroups in this section. Set $\widetilde{I}_0 = \widetilde{I} \cap \widetilde{K}_0$, and $\widetilde{T}_0 = \widetilde{T} \cap \widetilde{K}_0$. Observe that $\widetilde{T} \cap \widetilde{K}_0 \cong T \cap K$, viewed as a subgroup of $\widetilde{T}$, via Lemma 3.3.5. Moreover, set $N^- = \{(\mathrm{lt}(\varpi x), 1) \mid x \in \mathcal{O}\}$ and $N^+ = N \cap \widetilde{K}$. Observe that for $x, y \in \mathcal{O}$, $(\mathrm{lt}(\varpi x), 1)\,(\mathrm{lt}(\varpi y), 1) = \left(\mathrm{lt}(\varpi(x+y)), (\frac{x+y}{x}, \frac{x+y}{y})_n\right) = (\mathrm{lt}(\varpi(x+y)), 1)$, by part (vi) of Corollary 2.2.5. Hence, N^- is a subgroup of $\widetilde{G}$. After a quick calculation, it follows from Lemma 3.3.8 that

$$\widetilde{K}_0 = \widetilde{I}_0 \cup \widetilde{I}_0 \widetilde{w} \widetilde{I}_0, \text{ and } \widetilde{G} = \widetilde{B}\widetilde{I}_0 \cup \widetilde{B}\widetilde{w}\widetilde{I}_0. \tag{5.3.1}$$

We also make use of the Iwahori factorization

$$\widetilde{I}_0 = N^- \widetilde{T}_0 N^+. \tag{5.3.2}$$

Consider the subspace $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ of $(\pi_{\mathbf{s}})_N$. This subspace does not carry a representation of $\widetilde{T}$. However, since $\iota(\varpi^n)$ commutes with $\widetilde{T}_0$, the subspace $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ carries a representation of the subgroup $\langle \iota(\varpi^n), \widetilde{T}_0 \rangle$ of $\widetilde{T}$. Observe that

$$\langle \iota(\varpi^n), \widetilde{T}_0 \rangle = \{\iota(a) \mid a \in \mathbb{F}^\times, n|\mathrm{val}(a)\}.$$

In order to study the action of $\iota(\varpi^n)$ on $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$, we prove a variant of a well-known result due to Jacquet.

Proposition 5.3.1. *Let $f \in (\pi_{\mathbf{s}})^{\widetilde{T}_0 N^-}$. Then there exists a vector $f_0 \in (\pi_{\mathbf{s}})^{\widetilde{I}_0}$ such that f and f_0 have the same image in the Jacquet module $(\pi_{\mathbf{s}})_N$.*

Proof: In this proof we assume that the Haar measures are chosen such that $\widetilde{I}_0$, $\widetilde{T}_0 \cap N^-$, and N^+ all have measure equal to one. Set $f_0 = \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{x}) f d\mathtt{x}$. For $\mathtt{g} \in \widetilde{I}_0$

we have

$$\mathtt{g} \cdot \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{x}) f dx = \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{g})\pi_{\mathbf{s}}(\mathtt{x}) f d\mathtt{x} = \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{gx}) f d\mathtt{x}.$$

With the change of variable $\mathtt{gx} = \mathtt{x}'$, because $\widetilde{I}_0$ is unimodular, we have

$$\int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{x}') f d\mathtt{g}^{-1}\mathtt{x}' = \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{x}') f d\mathtt{x}' = f_0.$$

So $f_0 \in (\pi_{\mathbf{s}})^{\widetilde{I}_0}$. To show that f and f_0 have the same image in $(\pi_{\mathbf{s}})_N$, we show that $f - f_0 \in \pi_{\mathbf{s}}(N)$. We first show that $f_0 = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{x}) f d\mathtt{x}$. By the Iwahori factorisation 5.3.2, for all $\mathtt{x} \in \widetilde{I}_0$, $\mathtt{x}$ can be factored uniquely as $\mathtt{nm}$, with $\mathtt{n} \in N^+$, $\mathtt{m} \in \widetilde{T}_0 N^-$. So,

$$f_0 = \int_{N^+\widetilde{T}_0 N^-} \pi_{\mathbf{s}}(\mathtt{nm}) f d(\mathtt{nm}) = \int_{N^+} \int_{\widetilde{T}_0 N^-} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\mathtt{m}) f d\mathtt{n} d\mathtt{m},$$

which, because f is $\widetilde{T}_0 N^-$-fixed, equals

$$\int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}) f d\mathtt{n} \int_{\widetilde{T}_0 N^-} d\mathtt{m} = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}) f d\mathtt{n}.$$

Recall from Lemma 1.1.40 that a vector v is in $\pi_{\mathbf{s}}(N)$ if and only if $\int_{N_0} \pi_{\mathbf{s}}(\mathtt{n}) v d\mathtt{n} = 0$ for some compact subgroup N_0 of N. We calculate

$$\begin{aligned} \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})(f - f_0) d\mathtt{n} &= \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}) \left(f - \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}') f d\mathtt{n}' \right) d\mathtt{n} \\ &= \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}) f d\mathtt{n} - \int_{N^+} \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}') f d\mathtt{n}' d\mathtt{n} \\ &= \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}) f d\mathtt{n} - \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n}') f d\mathtt{n}' = 0. \end{aligned}$$

So, $f - f_0 \in \pi_{\mathbf{s}}(N)$, whence the result. ∎

Proposition 5.3.2. *Let J be the Jacquet functor. Then $J : (\pi_{\mathbf{s}})^{\widetilde{I}_0} \to ((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ is an isomorphism.*

Proof: Let J be the canonical map from $\pi_{\mathbf{s}}$ to $(\pi_{\mathbf{s}})_N$. Set $\bar{U} = ((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ and choose U to be a finite-dimensional subspace of $\pi_{\mathbf{s}}$ mapping onto $\bar{U}$ via J. We can assume that $U \subset (\pi_{\mathbf{s}})^{\widetilde{T}_0}$, and because $\pi_{\mathbf{s}}$ is smooth, there exists a compact open subgroup $N_\ell^- = \{(\mathrm{lt}(\varpi^\ell x), 1) \mid x \in \mathcal{O}\}$ such that $U \subseteq (\pi_{\mathbf{s}})^{\widetilde{T}_0 N_\ell^-}$. Moreover, choose $a \in \mathbb{F}^\times$ such that $\mathrm{val}(a) \geq \frac{\ell-1}{2}$ and $n|\mathrm{val}(a)$. Observe that for $\iota(a) \in \widetilde{T}$ and $(\mathrm{lt}(\varpi x), 1) \in N^-$,

$$\iota(a)^{-1}\,(\mathrm{lt}(\varpi x), 1)\,\iota(a) = \big(\mathrm{lt}(\varpi x a^2), 1\big) \tag{5.3.3}$$

lies in N_ℓ^-, by our choice of a. Now, we show that $\pi_{\mathbf{s}}(\iota(a))U \subseteq (\pi_{\mathbf{s}})^{\widetilde{T}_0 N^-}$. Let $\mathtt{n} \in N^-$, $\mathtt{b} \in \widetilde{T}_0$ and $u \in U \subseteq (\pi_{\mathbf{s}})^{\widetilde{T}_0 N_\ell^-}$. Then, by (5.3.3), there exists some $\mathtt{n}_1 \in N_\ell^-$ such that $\mathtt{n} = \iota(a)\mathtt{n}_1\iota(a)^{-1}$. Therefore,

$$\pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\iota(a))\,u = \pi_{\mathbf{s}}(\iota(a))\,\pi_{\mathbf{s}}(\mathtt{n}_1)\pi_{\mathbf{s}}\left(\iota(a)^{-1}\right)\pi_{\mathbf{s}}(\iota(a))\,u = \pi_{\mathbf{s}}(\iota(a))\,\pi_{\mathbf{s}}(\mathtt{n}_1)u = \pi_{\mathbf{s}}(\iota(a))\,u.$$

So, $\pi_{\mathbf{s}}(\iota(a))\,u$ is fixed by N^-. Moreover, observe that $\iota(a) \in A$, so it commutes with $\widetilde{T}_0$. Therefore, $\pi_{\mathbf{s}}(\mathtt{b})\pi_{\mathbf{s}}(\iota(a))\,u = \pi_{\mathbf{s}}(\iota(a))\,\pi_{\mathbf{s}}(\mathtt{b})u = \pi_{\mathbf{s}}(\iota(a))\,u$. Hence, $\pi_{\mathbf{s}}(\iota(a))U \subseteq (\pi_{\mathbf{s}})^{\widetilde{T}_0 N^-}$. Moreover, $J(\pi_{\mathbf{s}}(\iota(a))U) = (\pi_{\mathbf{s}})_N(\iota(a))\bar{U} = \bar{U}$, because $\iota(a)$ commutes with $\widetilde{T}_0$ and hence, $\bar{U}$ is invariant under $(\pi_{\mathbf{s}})_N(\iota(a))$. It follows that $J : \pi_{\mathbf{s}}(\iota(a))U \to \bar{U}$ is surjective. Because $\pi_{\mathbf{s}}(\iota(a))U \subseteq (\pi_{\mathbf{s}})^{\widetilde{T}_0 N^-}$, by Lemma 5.3.1 $J : (\pi_{\mathbf{s}})^{\widetilde{I}_0} \to ((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ is surjective.

To see that $J : \pi_{\mathbf{s}}^{\widetilde{I}_0} \to \pi_{\mathbf{s}N}^{\widetilde{T}_0}$ is injective, suppose $v \in \pi_{\mathbf{s}}^{\widetilde{I}_0}$ such that $J(v) = 0$. So $v \in \pi_{\mathbf{s}}^{\widetilde{I}_0} \cap \ker(J)$. Because $\pi_{\mathbf{s}}$ is admissible, $\pi_{\mathbf{s}}^{\widetilde{I}_0} \cap \ker(J)$ is finite-dimensional. Therefore, there exists an open compact subgroup $N_r = \{(\mathrm{ut}(x), 1) \mid \mathrm{val}(x) \geq r\} \subseteq N$ such that $\int_{N_r} \pi_{\mathbf{s}}(\mathtt{n})v\,d\mathtt{n} = 0$ for all $v \in \pi_{\mathbf{s}}^{\widetilde{I}_0} \cap \ker(J)$. Observe that for $a \in \mathbb{F}^\times$, with $\mathrm{val}(a) \geq -r/2$ we have $\iota(a)N_r\iota(a)^{-1} \subseteq N^+$. Moreover, if $\mathrm{val}(a) \geq 0$ and $n|\mathrm{val}(a)$, it is not difficult to see that $\pi_s(\iota(a))\left(\pi_{\mathbf{s}}^{\widetilde{I}_0}\right) \subseteq \pi_{\mathbf{s}}^{\widetilde{T}_0 N^-}$. Choose $a \in \mathbb{F}^\times$ such that both the above conditions are satisfied. Note that if N_* is an open compact subgroup of N such that $N_r \subset N_*$, $\int_{N_*} \pi_{\mathbf{s}}(\mathtt{n})v\,d\mathtt{n} = 0$, for all $v \in \pi_{\mathbf{s}}^{\widetilde{I}_0} \cap \ker(J)$, so we can choose such an open compact subgroup so that $\iota(a)N_*\iota(a)^{-1} = N^+$.

Set $\Omega := \widetilde{I}_0\iota(a)\widetilde{I}_0$. Let ch_Ω be the $\widetilde{I}_0$-bi-invariant characteristic function of Ω, so $\mathrm{ch}_\Omega(\iota(a)) = 1$. It can be checked that $\mathrm{Im}(\pi_{\mathbf{s}}(\mathrm{ch}_\Omega)) = \pi_{\mathbf{s}}^{\widetilde{I}_0}$ and that the operator $\pi_{\mathbf{s}}(\mathrm{ch}_\Omega)$ on $\pi_{\mathbf{s}}^{\widetilde{I}_0}$ is invertible [Cas95, Proposition 9.2.3]. Therefore, there exists $v' \in \pi_{\mathbf{s}}^{\widetilde{I}_0}$

such that $v = \pi_{\mathbf{s}}(\mathrm{ch}_\Omega)v'$; that is, $v = \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{k})\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{k}$. Observe that by our choice of a, $\pi_{\mathbf{s}}(\iota(a))v' \in \pi_{\mathbf{s}}^{\widetilde{T}_0 N^-}$. Therefore, by the argument in the proof of Proposition 5.3.1

$$v = \int_{\widetilde{I}_0} \pi_{\mathbf{s}}(\mathtt{k})\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{k} = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{n}.$$

Hence,

$$\begin{aligned} 0 = \int_{N_*} \pi_{\mathbf{s}}(\mathtt{n}) v d\mathtt{n} &= \int_{N_*}\int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\mathtt{n}')\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{n}' d\mathtt{n} \\ &= \int_{\iota(a)^{-1}N^+\iota(a)}\int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\mathtt{n}')\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{n}' d\mathtt{n}, \end{aligned}$$

which, after renormalizing the measure suitably throughout, is equal to

$$\pi_{\mathbf{s}}(\iota(a)) \int_{\iota(a)^{-2}N^+\iota(a)^2}\int_{\iota(a)^{-1}N^+\iota(a)} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\mathtt{n}')v' d\mathtt{n}' d\mathtt{n} = \int_{\iota(a)^{-2}N^+\iota(a)^2} \pi_{\mathbf{s}}(\mathtt{n})v' d\mathtt{n}.$$

In the last step, we use the fact that $\iota(a)^{-1}N^+\iota(a) \subseteq \iota(a)^{-2}N^+\iota(a)^2$ are subgroups of the commutative group N, and therefore, we can change the order of the integrals.

Hence, $J(v') = 0$, and because $v' \in \pi_{\mathbf{s}}^{\widetilde{I}_0}$, it follows that $\int_{\iota(a)^{-1}N^+\iota(a)} \pi_{\mathbf{s}}(\mathtt{n})v' d\mathtt{n} = 0$. Therefore,

$$\pi_{\mathbf{s}}(\iota(a)^{-1}) \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{n} = 0,$$

which implies that $v = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}(\iota(a))v' d\mathtt{n} = 0$. ∎

For every $\mathtt{g} \in \widetilde{G}$, write $\mathtt{g} = \mathtt{bk}$ according to its Iwasawa decomposition. Define the functions f_1 and $f_{\widetilde{w}}$ in $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$ as follows:

$$f_1(\mathtt{bk}) = \begin{cases} \rho_{\mathbf{s}}(\mathtt{b})f_0, & \mathtt{k} \in \widetilde{I}_0; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\widetilde{w}}(\mathtt{bk}) = \begin{cases} \rho_{\mathbf{s}}(\mathtt{b})f_0, & \mathtt{k} \in \widetilde{I}_0\widetilde{w}\widetilde{I}_0; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.3.3. *The set $\{f_1, f_{\widetilde{w}}\}$ is a basis for $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$.*

Proof: Evidently, f_1 and $f_{\widetilde{w}}$ are independent elements of $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$. It follows from the decomposition $\widetilde{G} = \widetilde{B}\widetilde{I}_0 \cup \widetilde{B}\widetilde{w}\widetilde{I}_0$ that functions in $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$ are determined completely by their values on 1 and $\widetilde{w}$. ∎

Because $\widetilde{T}_0$ acts trivially on $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$, to study the action of the subgroup $\langle \iota(\varpi^n), \widetilde{T}_0 \rangle$ on $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$, it is enough to study the action of the element $\iota(\varpi^n)$. The isomorphism $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0} \cong (\pi_{\mathbf{s}})^{\widetilde{I}_0}$ in Proposition 5.3.2 suggests considering the action of $\iota(\varpi^n)$ on the basis elements of $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$. But $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$ is not invariant under the action of $\iota(\varpi^n)$; in particular, $\iota(\varpi^n)\cdot f_1$ and $\iota(\varpi^n)\cdot f_{\widetilde{w}}$ are not $\widetilde{I}_0$-fixed. However, we observe that $\iota(\varpi^n)\cdot f_1$ and $\iota(\varpi^n)\cdot f_{\widetilde{w}}$ are $\widetilde{T}_0 N^-$-fixed. Hence, by Lemma 5.3.1, there exists $\overline{\iota(\varpi^n)\cdot f_1}$ and $\overline{\iota(\varpi^n)\cdot f_{\widetilde{w}}}$ in $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$, such that $\overline{\iota(\varpi^n)\cdot f_1}$ and $\overline{\iota(\varpi^n)\cdot f_{\widetilde{w}}}$ have the same image under the Jacquet functor as $\iota(\varpi^n)\cdot f_1$ and $\iota(\varpi^n)\cdot f_{\widetilde{w}}$ respectively. In the next proposition, we use this idea to calculate the action of $\iota(\varpi^n)$ on $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$.

Proposition 5.3.4. *The matrix of the action of $\iota(\varpi^n)$ on $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ with respect to the basis $\{J(f_1), J(f_{\widetilde{w}})\}$ is*

$$\begin{pmatrix} q^{-n\mathbf{s}} & (q-1)q^{-n\mathbf{s}-1}(1+q^{n\mathbf{s}-n}) \\ 0 & q^{n\mathbf{s}-2n} \end{pmatrix}.$$

Proof: Let $\{f_1, f_{\widetilde{w}}\}$ be the basis for $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$ given in Lemma 5.3.3. Hence, by Proposition 5.3.1, $\{J(f_1), J(f_{\widetilde{w}})\}$ is a basis for $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$. Consider $\iota(\varpi^n)\cdot f_1$ and $\iota(\varpi^n)\cdot f_{\widetilde{w}}$ in $\pi_{\mathbf{s}}$. Note that $\iota(\varpi^n)$ normalizes N^- and commutes with $\widetilde{T}_0$, so that $\iota(\varpi^n)\cdot f_1$ and $\iota(\varpi^n)\cdot f_{\widetilde{w}}$ are $\widetilde{T}_0 N^-$-fixed. Therefore, by Proposition 5.3.1, $\overline{\iota(\varpi^n)\cdot f_\alpha} = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}\left(\iota(\varpi^n)\right) f_\alpha d\mathtt{n}$ are $\widetilde{I}_0$-fixed, and have the same image in $(\pi_{\mathbf{s}})_N$ as $\iota(\varpi^n)\cdot f_\alpha$, $\alpha \in \{1, \widetilde{w}\}$ (up to a scalar). We deduce that, $\iota(\varpi^n)\cdot J(f_\alpha) = J(\overline{\iota(\varpi^n)\cdot f_\alpha})$. Identifying $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ and $(\pi_{\mathbf{s}})^{\widetilde{I}_0}$ via J, we calculate $\overline{\iota(\varpi^n)\cdot f_\alpha}$, $\alpha \in \{1, \widetilde{w}\}$.

Since every function in $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ is completely determined by its values on $(\mathrm{I}_2, 1)$ and $\widetilde{w}$, we calculate the values of $\overline{\iota(\varpi^n)\cdot f_\alpha}$ at $(\mathrm{I}_2, 1)$ and $\widetilde{w}$. Recall that the Haar measure on $\mathbb{F}$ is normalized so that $\mu(\mathcal{O}) = 1$.

First, we calculate $\overline{\iota(\varpi^n)\cdot f_\alpha}(\mathrm{I}_2, 1) = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}\left(\iota(\varpi^n)\right) f_\alpha(\mathrm{I}_2, 1) d\mathtt{n}$. Note that, for $\alpha \in \{1, \widetilde{w}\}$,

$$\pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}\left(\iota(\varpi^n)\right) f_\alpha(\mathrm{I}_2, 1) = f_\alpha\left((\mathrm{ut}(m), 1)\iota(\varpi^n)\right), \tag{5.3.4}$$

where $\mathtt{n} = (\mathrm{ut}(m), 1)$, for some $m \in \mathcal{O}$. Note that $(\mathrm{ut}(m), 1)\iota(\varpi^n) \in \widetilde{B} \subset \widetilde{B}\widetilde{I}_0$. Hence, by the definition of f_α, (5.3.4) is zero if $\alpha = \widetilde{w}$, and is equal to $\rho_{\mathbf{s}}\left((\mathrm{ut}(m), 1)\iota(\varpi^n)\right) f_0$ if $\alpha = 1$. Observe that, in the latter case, because $\iota(\varpi^n) \in Z(\widetilde{T})$, and $\rho_{\mathbf{s}}$ is trivial on N, $\rho_{\mathbf{s}}\left((\mathrm{ut}(m), 1)\iota(\varpi^n)\right) f_0 = \chi_{\mathbf{s}}\left(\iota(\varpi^n)\right) f_0 = q^{-n\mathbf{s}} f_0$. Hence,

$$\overline{\iota(\varpi^n) \cdot f_\alpha}(\mathrm{I}_2, 1) = \int_{\mathcal{O}} f_\alpha\left((\mathrm{ut}(m), 1)\iota(\varpi^n)\right) d\, m = \begin{cases} q^{-n\mathbf{s}} f_0, & \alpha = 1 \\ 0, & \alpha = \widetilde{w}. \end{cases} \tag{5.3.5}$$

Next, we calculate $\overline{\iota(\varpi^n) \cdot f_\alpha}(\widetilde{w}) = \int_{N^+} \pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}\left(\iota(\varpi^n)\right) f_\alpha(\widetilde{w}) d\mathtt{n}$. Observe that,

$$\pi_{\mathbf{s}}(\mathtt{n})\pi_{\mathbf{s}}\left(\iota(\varpi^n)\right) f_\alpha(\widetilde{w}) = f_\alpha(\widetilde{w}(\mathrm{ut}(m), 1)\iota(\varpi^n)) = f_\alpha\left(\iota(\varpi^{-n})\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}), 1)\right), \tag{5.3.6}$$

where $\mathtt{n} = (\mathrm{ut}(m), 1)$, for some $m \in \mathcal{O}$, and $\alpha \in \{1, \widetilde{w}\}$. We are in one of the following situations.

If $\mathrm{val}(m) \geq 2n$, then $m\varpi^{-2n} \in \mathcal{O}$, so that $(\mathrm{ut}(m\varpi^{-2n}), 1) \in N^+$, yielding $\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}), 1) \in \widetilde{w}\widetilde{I}_0$. Therefore, in this case

$$f_\alpha\left(\iota(\varpi^{-n})\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}), 1)\right) = \begin{cases} 0, & \alpha = 1 \\ \rho_{\mathbf{s}}(\iota(\varpi^{-n})) f_0, & \alpha = \widetilde{w}. \end{cases} \tag{5.3.7}$$

Note that $\rho_{\mathbf{s}}(\iota(\varpi^{-n})) f_0 = \chi_{\mathbf{s}}(\iota(\varpi^{-n})) f_0 = q^{n\mathbf{s}} f_0$.

If $0 \leq \mathrm{val}(m) < 2n$ then it is not difficult to see that we can write

$$\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}), 1) = \left(\begin{pmatrix} -\varpi^{2n}m^{-1} & 1 \\ 0 & -\varpi^{-2n}m \end{pmatrix}, 1\right)\left(\mathrm{lt}(\varpi^{2n}m^{-1}), 1\right).$$

Observe that $\left(\left(\begin{smallmatrix} -\varpi^{2n}m^{-1} & 1 \\ 0 & -\varpi^{2n}m \end{smallmatrix}\right), 1\right) \in \widetilde{B}$, and because $0 \leq \mathrm{val}(m) < 2n$, $(\mathrm{lt}(\varpi^{2n}m^{-1}), 1) \in N^- \subset \widetilde{I}_0$. Hence, in this case, $f_\alpha\left(\iota(\varpi^{-n})\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}), 1)\right)$ equals

$$\rho_{\mathbf{s}}\left(\iota(\varpi^{-n})\left(\begin{pmatrix} -\varpi^{2n}m^{-1} & 1 \\ 0 & -\varpi^{-2n}m \end{pmatrix}, 1\right)\right) f_\alpha\left(\mathrm{lt}(\varpi^{2n}m^{-1}), 1\right),$$

which, by the definition of f_α, equals

$$\begin{cases} \rho_{\mathbf{s}}\left(\iota(\varpi^{-n})\right)\rho_{\mathbf{s}}\left(\left(\begin{smallmatrix} -\varpi^{2n}m^{-1} & 1 \\ 0 & -\varpi^{-2n}m \end{smallmatrix}\right),1\right) f_0, & \alpha = 1 \\ 0, & \alpha = \widetilde{w}. \end{cases} \tag{5.3.8}$$

Also, after a quick calculation, we can write,

$$\rho_{\mathbf{s}}\left(\left(\begin{smallmatrix} -\varpi^{2n}m^{-1} & 1 \\ 0 & -\varpi^{-2n}m \end{smallmatrix}\right),1\right) = \rho_{\mathbf{s}}\left(\left(\mathrm{ut}(-\varpi^{2n}m^{-1}),1\right)\iota(-\varpi^{2n})\iota(m^{-1})\right),$$

which because $\rho_{\mathbf{s}}$ is trivial on N and $\iota(-\varpi^{2n}) \in Z(\widetilde{T})$ equals $q^{-2ns}\rho_{\mathbf{s}}(\iota(m^{-1}))$.

Let $U = \{m \in \mathcal{O} \mid 0 \leq \mathrm{val}(m) < 2n\}$. It follows from (5.3.6) that $\overline{\iota(\varpi^n) \cdot f_\alpha}(\widetilde{w})$ is equal to

$$\int_{\varpi^{2n}\mathcal{O}} f_\alpha\left(\iota(\varpi^{-n})\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}),1)\right)\,dm + \int_U f_\alpha\left(\iota(\varpi^{-n})\widetilde{w}(\mathrm{ut}(m\varpi^{-2n}),1)\right)\,dm.$$

Suppose $\alpha = \widetilde{w}$. Then, by (5.3.7) and (5.3.8), because $\mu(\varpi^{2n}\mathcal{O}) = q^{-2n}$,

$$\overline{\iota(\varpi^n)\cdot f_{\widetilde{w}}}(\widetilde{w}) = \int_{\varpi^{2n}\mathcal{O}} q^{n\mathbf{s}} f_0 dm = q^{n\mathbf{s}-2n} f_0.$$

Suppose $\alpha = 1$. Let $U_i = \{x \in \mathcal{O} \mid \mathrm{val}(m) = i\}$. Then by (5.3.7) and (5.3.8),

$$\overline{\iota(\varpi^n)\cdot f_1}(\widetilde{w}) = \int_U q^{-n\mathbf{s}}\rho_{\mathbf{s}}(\iota(m^{-1}))f_0\,dm = q^{-n\mathbf{s}}\sum_{i=0}^{2n-1}\int_{U_i}\rho_{\mathbf{s}}(\iota(m^{-1}))f_0\,d_m.$$

For $m \in U_i$, write $m = \varpi^i a$ for some $a \in \mathcal{O}^\times$. Hence,

$$\rho_{\mathbf{s}}(\iota(m^{-1}))f_0 = \rho_{\mathbf{s}}(\iota(\varpi^{-i}))\rho_{\mathbf{s}}(\mathrm{dg}(a^{-1}),\vartheta(m))f_0.$$

Recall from Lemma 5.1.5 that $\rho_{\mathbf{s}}(\mathrm{dg}(a^{-1}),\vartheta(m))f_0 = \epsilon(\vartheta(m))f_0$, hence $\overline{\iota(\varpi^n)\cdot f_1}(\widetilde{w})$ simplifies to

$$q^{-n\mathbf{s}}\sum_{i=0}^{2n-1}\rho_{\mathbf{s}}(\iota(\varpi^{-i}))\int_{U_i}\epsilon(\vartheta(m))f_0\,dm.$$

Observe that for a fixed i, $\vartheta(m) = \vartheta^i_{\mathcal{O}^\times}(a)$, where $m = \varpi^i a$. Also, note that this latter integral is non-zero if and only if ϑ is trivial on U_i; that is, $\vartheta^i_{\mathcal{O}^\times}$ is trivial; that is, $n|i$, i.e., for $i = 0$ or $i = n$. Observe that $\rho_{\mathbf{s}}(\iota(\varpi^{-i}))f_0$ is f_0 if $i = 0$, and is $q^{n\mathbf{s}} f_0$ if $i = n$. Observe that $\mu(U_0) = (q-1)q^{-1}$ and $\mu(U_n) = (q-1)q^{-1}q^{-n}$. Therefore,

$$\overline{\iota(\varpi^n) \cdot f_1}(\widetilde{w}) = q^{-n\mathbf{s}}\left[(q-1)q^{-1}f_0 + (q-1)q^{-1}q^{-n}q^{n\mathbf{s}}f_0\right] = (q-1)(q^{-n\mathbf{s}-1})(1+q^{n\mathbf{s}-n})f_0.$$

∎

Corollary 5.3.5. *Let $\chi_{\mathbf{s}}$ be a non-regular character of $Z(\widetilde{T})$. The representation $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ of $\langle \iota(\varpi^n), \widetilde{T}_0 \rangle$ is indecomposable for $\mathbf{s} = 1$ and completely decomposable for $\mathbf{s} = 1 + \pi i/\log q$.*

Proof: The result follows from specializing the matrix in Proposition 5.3.4 to the non-regular cases, $\mathbf{s} = 1$ and $\mathbf{s} = 1 + \pi i/\log q$. The matrix of the action of $\iota(\varpi^n)$ on $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ with respect to the basis $\{f_1, f_{\widetilde{w}}\}$ for $\mathbf{s} = 1$ is

$$\begin{pmatrix} q^{-n} & 2(q-1)q^{-(n+1)} \\ 0 & q^{-n} \end{pmatrix},$$

which is a non-diagonalizable matrix, and for $\mathbf{s} = 1 + \pi i/\log q$, we get the diagonal matrix

$$\begin{pmatrix} -q^{-n} & 0 \\ 0 & -q^{-n} \end{pmatrix}.$$

∎

We summarize the result in the following Theorem.

Theorem 5.3.6. *The principal series representation $\pi_{\mathbf{s}}$ is irreducible for $\mathbf{s} = 1$ and decomposes into two inequivalent sub-representations for $\mathbf{s} = 1 + \pi i/\log q$.*

Proof: Suppose $\mathbf{s} = 1$. Then $\chi_{\mathbf{s}}$ is a non-regular character, so that $\rho_{\mathbf{s}} \cong \rho_{-\mathbf{s}+2}$. Hence the exact sequence in Lemma 5.2.1 modifies to

$$0 \to \rho_{\mathbf{s}} \to (\pi_{\mathbf{s}})_N \to \rho_{\mathbf{s}} \to 0.$$

If $(\pi_{\mathbf{s}})_N$ were decomposable, it would decompose as $\rho_{\mathbf{s}}\oplus\rho_{\mathbf{s}}$. Consequently, $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0} = \rho_{\mathbf{s}}^{\widetilde{T}_0}\oplus\rho_{\mathbf{s}}^{\widetilde{T}_0}$. But, Corollary 5.3.5 implies that $((\pi_{\mathbf{s}})_N)^{\widetilde{T}_0}$ is indecomposable. Hence, $(\pi_{\mathbf{s}})_N$ is indecomposable. Therefore, up to a scalar, there is only one intertwining map from $(\pi_{\mathbf{s}})_N$ to $\rho_{\mathbf{s}}$. By Lemma 1.1.41,

$$\dim \mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{\mathbf{s}}) \quad = \quad \dim \mathrm{Hom}_{\widetilde{T}}((\pi_{\mathbf{s}})_N, \rho_{\mathbf{s}}) = 1.$$

Since $\pi_{\mathbf{s}}$ is unitary, $\dim \mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{\mathbf{s}}) = 1$ if and only if $\pi_{\mathbf{s}}$ is irreducible.

Suppose $\mathbf{s} = 1+\pi i/\log q$. We explicitly produce subspaces of $(\pi_{\mathbf{s}})_N$ isomorphic to $\rho_{\mathbf{s}}$. Let us drop the J from the notation. Consider the spaces $U_\alpha = \mathrm{Span}\{\iota(\varpi)^i\cdot f_\alpha | 0 \leq i < n\}$ for $\alpha \in \{1, \widetilde{w}\}$. Observe that $\widetilde{T} = \langle \widetilde{T}_0, \{\mathrm{I}_2\}\times\mu_n, \iota(\varpi)\rangle$. Note that since the f_α are $\widetilde{T}_0$-fixed, $\widetilde{T}_0$ acts trivially on elements of U_α, and that $\{1\}\times\mu_n$ acts on elements of U_α by the faithful character ϵ. So, for every $\iota(t)\in\widetilde{T}_0$ and $0\leq i<n$,

$$(\iota(t)\iota(\varpi^i))\cdot f_\alpha = \epsilon((\varpi^i, t)_n)\iota(\varpi^i)\iota(t)\cdot f_\alpha = \epsilon((\varpi^i, t)_n)\iota(\varpi^i)\cdot f_\alpha.$$

Furthermore, because by Corollary 5.3.5 $\iota(\varpi^n)\cdot f_\alpha = -q^{-n}f_\alpha$, $\alpha\in\{1,\widetilde{w}\}$,

$$\iota(\varpi)\iota(\varpi^i)\cdot f_\alpha = \begin{cases} \iota(\varpi^{i+1})\cdot f_\alpha, & i < n-1; \\ -q^{-n}f_\alpha, & i = n-1. \end{cases}$$

Hence, the U_α are $\widetilde{T}$-invariant; also the action $\iota(\varpi)$ shows that the U_α are irreducible. Therefore, by the Stone-von Neumann theorem, $U_\alpha \cong \mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$, and hence is n-dimensional. Also, note that $U_1\cap U_{\widetilde{w}} = \{1\}$. Hence, by dimension comparison, $(\pi_{\mathbf{s}})_N = U_1\oplus U_{\widetilde{w}} \cong \rho_{\mathbf{s}}^{\oplus 2}$. So,

$$\dim \mathrm{Hom}_{\widetilde{T}}((\pi_{\mathbf{s}})_N, \rho_{\mathbf{s}}) \quad = \quad \dim \mathrm{Hom}_{\widetilde{T}}(\rho_{\mathbf{s}}\oplus\rho_{\mathbf{s}}, \rho_{\mathbf{s}}) = 2.$$

By Lemma 1.1.41, $\dim \mathrm{Hom}_{\widetilde{G}}(\pi_{\mathbf{s}}, \pi_{\mathbf{s}}) = \dim \mathrm{Hom}_{\widetilde{T}}((\pi_{\mathbf{s}})_N, \rho_{\mathbf{s}}) = 2$. Hence, because $\pi_{\mathbf{s}}$ is unitary, it decomposes into a direct sum of two inequivalent subspaces. ∎

Chapter 6

On the K-Type Distribution of Unitary Unramified Principal Series Representations of $\widetilde{\mathrm{SL}}_2(\mathbb{F})$

In this Chapter, we investigate the problem of the distribution of the $\widetilde{K}$-irreducible spaces in the K-type decomposition in Theorem 4.2.13, into the $\widetilde{G}$-irreducible constituents of $\pi_{\mathbf{s}}$ in Theorem 5.3.6, when $\pi_{\mathbf{s}}$ is a reducible unramified unitary representation. My approach, inspired by the argument in [GGPS90] for the linear group $\mathrm{SL}_2(\mathbb{F})$, is to realize the principal series representation and its K-types as subspaces of $L^2(\mathbb{F}, \mathbb{C}^n)$. Then we realize the two irreducible components of $\pi_{\mathbf{s}}$, in Theorem 5.3.6, as kernels of intertwining operators from $\pi_{\mathbf{s}}$ to itself, and investigate when a function in a certain K-type belongs to such kernels. In this chapter, following [GGPS90], it is more convenient to use normalized induction, defined in Remark 1.1.30, to construct the principal series representations of $\widetilde{G}$. We give the correspondence between the two types of induction in Remark 6.1.2.

Our main results, stated in Theorem 6.2.3 and Theorem 6.2.6, show that the two irreducible components of $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0\right)^{K_1}$ belong to opposite subrepresentations of $\pi_{\mathbf{s}}$.

Throughout this Chapter, we assume that $n > 2$ is an odd integer which divides $q-1$. Recall that $\mathrm{dg}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$, $\iota(t) = (\mathrm{dg}(t), 1)$ for all $t \in \mathbb{F}^{\times}$, and $\mathrm{ut}(m) = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$, $\mathrm{lt}(m) = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$ for all $m \in \mathbb{F}$, and $\widetilde{w} = (w, 1) = (\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), 1)$. Note that in general $\iota(t_1)\iota(t_2) = \iota(t_1 t_2)\epsilon((t_1, t_2)_n)$, for $t_1, t_2 \in \mathbb{F}^{\times}$. However, because n is odd, $(\varpi, \varpi)_n = 1$ so that $\iota(\varpi)^k = \iota(\varpi^k)$, for every integer k.

For every positive integer m, let $\mathrm{rsd}_m(k)$ be the least non-negative residue of k mod m. As in Chapter 1, fix an additive character $\lambda : \mathbb{F} \to \mathbb{C}^{\times}$ such that $\lambda|_{\mathcal{O}} = 1$ and $\lambda|_{\mathfrak{p}^{-1}} \neq 1$. Note that λ is a unitary character. Recall that $\vartheta : \mathbb{F}^{\times} \to \mu_n$ given via $a \mapsto (\varpi, a)_n$ is a ramified character of degree one.

6.1 λ-Realization of the Principal Series Representations of $\widetilde{G}$

As in Chapter 5, for each $\mathbf{s} \in \mathbb{C}$, let $\chi_{\mathbf{s}}$ be the unitary unramified character of $Z(\widetilde{T})$ defined by

$$\begin{array}{rcl} \chi_{\mathbf{s}} : Z(\widetilde{T}) & \to & \mathbb{C} \\ (\mathrm{dg}(t), \zeta) & \mapsto & |t|^{\mathbf{s}} \epsilon(\zeta), \end{array}$$

extended (trivially) to a maximal abelian subgroup A of $\widetilde{T}$, and let $(\rho_{\mathbf{s}}, \mathrm{Ind}_A^{\widetilde{T}} \chi_{\mathbf{s}})$ be the corresponding unique (up to isomorphism) genuine irreducible representation of $\widetilde{T}$, obtained by applying the Stone-von Neumann theorem.

Let f_i, for $0 \leq i < n$, be a function in $\mathrm{Ind}_A^{\widetilde{T}} \chi_{\mathbf{s}}$ whose support is in the coset $A\iota(\varpi^i)$, so that $f_i(\iota(\varpi^i)) = 1$. We fix the basis $\{f_0, \ldots, f_{n-1}\}$ for $\mathrm{Ind}_A^{\widetilde{T}} \chi_{\mathbf{s}}$. We identify the space $\mathrm{Ind}_A^{\widetilde{T}} \chi_{\mathbf{s}}$ with $\mathbb{C}^n$ via the following map:

$$f \quad \mapsto \quad (f(1), f(\iota(\varpi)), \ldots, f(\iota(\varpi)^{n-1})). \tag{6.1.1}$$

The basis $\{f_0, \ldots, f_{n-1}\}$ is mapped to the standard basis of $\mathbb{C}^n$ under the above identification. We write the action of $\widetilde{T}$ as matrices with respect to this basis. Since $\widetilde{T}$ is generated by A and $\iota(\varpi)$, this action is completely determined by $\rho_{\mathbf{s}}(\mathsf{a})$, $\mathsf{a} \in A$ and $\rho_{\mathbf{s}}(\iota(\varpi))$. Recall that $A = \{(\mathrm{dg}(a), \zeta) \mid a \in \mathbb{F}^{\times}, n|\mathrm{val}(a), \zeta \in \mu_n\}$.

Lemma 6.1.1. *Let* $\mathsf{a} := (\mathrm{dg}(a), \zeta) \in A$*, and* $\iota(\varpi) \in \widetilde{T}$*. Then*

$$\rho_{\mathbf{s}}(\iota(\varpi)) = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 \\ |\varpi^n|^{\mathbf{s}} & \ldots & 0 & 0 & 0 \end{pmatrix}, \tag{6.1.2}$$

and

$$\rho_{\mathbf{s}}(\mathsf{a}) = |a|^{\mathbf{s}} \epsilon(\zeta) \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \epsilon\left(\vartheta(a)^{-2}\right) & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \epsilon\left(\vartheta(a)^{-2(n-1)}\right) \end{pmatrix}. \tag{6.1.3}$$

Proof: For any $f \in \mathrm{Ind}_A^{\widetilde{T}} \chi_{\mathbf{s}}$, the map (6.1.1) takes $\iota(\varpi) \cdot f$ to

$$\begin{aligned} \left(\iota(\varpi) \cdot f(1), \ldots, \iota(\varpi) \cdot f(\iota(\varpi^{n-1}))\right) &= \left(f(\iota(\varpi)), \ldots, f(\iota(\varpi)^{n-1}), f(\iota(\varpi)^n)\right) \\ &= \left(f(\iota(\varpi)), \ldots, f(\iota(\varpi)^{n-1}), |\varpi^n|^s f(1)\right). \end{aligned}$$

The last equality uses the fact that $\iota(\varpi)^n \in A$, and hence $f(\iota(\varpi)^n) = \chi_{\mathbf{s}}(\iota(\varpi)^n) f(1)$. Hence, for any $(z_0, \ldots, z_{n-1}) \in \mathbb{C}^n$

$$\iota(\varpi) \cdot (z_0, \ldots, z_{n-1}) = (z_1, z_2, \ldots, z_{n-1}, |\varpi^n|^{\mathbf{s}} z_0),$$

which implies the matrix (6.1.2).

Now, suppose $\mathsf{a} = (\mathrm{dg}(a), \zeta) \in A$. Then

$$\iota(\varpi^i)\mathsf{a} = \left(\mathrm{dg}(a), \vartheta(a)^{-2i}\zeta\right) \iota(\varpi^i).$$

Hence, the map (6.1.1) sends $\mathtt{a}\cdot f$ to

$$\begin{aligned}(\mathtt{a}\cdot f(1),\ldots,\mathtt{a}\cdot f(\iota(\varpi^{n-1}))) &= (f(\mathtt{a}),\ldots,f(\iota(\varpi)^{n-1}\mathtt{a}))\\ &= |a|^{\mathbf{s}}\epsilon(\zeta)\left(f(1),\ldots,\epsilon\left(\vartheta(a)^{-2(n-1)}\right)f(\iota(\varpi^{n-1}))\right).\end{aligned}$$

So, the action of $\mathtt{a}$ on $\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}$ with respect to our choice of basis is

$$\mathtt{a}\cdot(z_0,\ldots,z_{n-1}) = |a|^s\epsilon(\zeta)\left(z_0,\epsilon\left(\vartheta(a)^{-2}\right)z_1,\ldots,\epsilon\left(\vartheta(a)^{-2(n-1)}\right)z_{n-1}\right),$$

which gives the matrix (6.1.3). ∎

Recall that for $\mathtt{b} = \left(\left(\begin{smallmatrix} t & n \\ 0 & t^{-1}\end{smallmatrix}\right),\zeta\right) \in \widetilde{B}$, the modular character is $\delta_{\tilde{B}}(\mathtt{b}) = |t|^2$. Let $(\bar{\pi}_{\mathbf{s}}, \mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}})$ denote the normalized principal series representation of $\widetilde{G}$. That is,

$$\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}} = \{f:\widetilde{G}\to\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}\ :\ f(\mathtt{bg}) = \delta_{\tilde{B}}^{1/2}(\mathtt{b})\rho_{\mathbf{s}}(\mathtt{b})f(\mathtt{g}),\ \mathtt{b}\in\widetilde{B},\mathtt{g}\in\widetilde{G}\},$$

where $\mathtt{b} = \left(\left(\begin{smallmatrix} t & n \\ 0 & t^{-1}\end{smallmatrix}\right),\zeta\right)\in\widetilde{B}$ and $\mathtt{g}\in\widetilde{G}$, and $\bar{\pi}_{\mathbf{s}}$ acts by right translation.

Remark 6.1.2. Note that, by definition, $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}} = \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\delta_{\tilde{B}}^{1/2}\rho_{\mathbf{s}}$, where "Ind" denotes the non-normalized induction we considered in Chapter 5 and Chapter 4. Also, similar to the proof of Lemma 5.1.12, it is not difficult to see that $(\delta_{\tilde{B}}^{1/2}\rho_{\mathbf{s}}, \mathcal{E}(\mathrm{Ind}_A^{\widetilde{T}}\chi_{\mathbf{s}}))$ and $(\rho_{\mathbf{s}+1}, \mathrm{Ind}_A^{\widetilde{T}}\chi_{-\mathbf{s}+2})$ have the same central characters and therefore, by the Stone-von Neumann theorem, are isomorphic. Hence, $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}} \cong \mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\mathbf{s}+1}$.

It follows from Theorem 5.3.6 and Remark 6.1.2 that $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\frac{\pi i}{\log q}+1} \cong \mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\frac{\pi i}{\log q}}$ decomposes into two irreducible factors. In this chapter, we investigate the distribution of K-types of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho_{\frac{\pi i}{\log q}}$ into these two irreducible components.

Remark 6.1.3. For the rest of this chapter, we set $\mathbf{s} = \frac{\pi i}{\log q}$ and drop the subscript $\mathbf{s}$ from the notation of χ and ρ. Note that under this assumption

$$|x|^{\mathbf{s}} = (-1)^{\mathrm{val}(x)} \text{ for all } x\in\mathbb{F}^{\times},$$

and from (6.1.2), it is not difficult to see that

$$\rho_{i,j}\left(\iota(\varpi^{-k})\right) = \begin{cases} 1, & i-j=k \\ -1, & j-i=n-k \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.4}$$

6.1.1 The λ-Realization of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$

We continue identifying $\mathrm{Ind}_A^{\widetilde{T}}\chi$ with $\mathbb{C}^n$ (6.1.1). Consider the map

$$\begin{aligned} \boldsymbol{\eta} : \mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho &\to L^2(\mathbb{F}, \mathbb{C}^n) \\ f &\mapsto \eta_f, \end{aligned} \tag{6.1.5}$$

where

$$\eta_f(x) := f\left(\mathrm{lt}(x), 1\right).$$

The next lemma gives the induced action of $\widetilde{G}$ on $L^2(\mathbb{F}, \mathbb{C}^n)$.

Lemma 6.1.4. *Let δ and $\gamma \in \mathbb{F}^\times$, and $\zeta \in \mu_n$. The action of $\widetilde{G}$ on $\boldsymbol{\eta}\left(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho\right) \subseteq L^2(\mathbb{F}, \mathbb{C}^n)$ is completely determined by the following formulas:*

$$(\mathrm{dg}(\delta), \zeta) \cdot \eta_f(x) = \epsilon(\zeta)|\delta|\rho(\iota(\delta))\eta_f(\delta^2 x), \tag{6.1.6}$$

$$(\mathrm{lt}(\gamma), 1) \cdot \eta_f(x) = \eta_f(x+\gamma), \tag{6.1.7}$$

and

$$\widetilde{w} \cdot \eta_f(x) = |x^{-1}|\rho(\iota(x^{-1}))\eta_f(-x^{-1}), \tag{6.1.8}$$

for every $x \in \mathbb{F}^\times$.

Proof: It follows from Bruhat decomposition in Lemma 3.3.8, and the fact that $\mathrm{ut}(x) = \widetilde{w}\mathrm{lt}(-x)\widetilde{w}^{-1}$ for all $x \in \mathbb{F}$, that $\widetilde{G}$ is generated by matrices

$$(\mathrm{dg}(\delta), \zeta), \quad (\mathrm{lt}(\gamma), 1), \quad \widetilde{w},$$

where $\delta \in \mathbb{F}^\times$, $\gamma \in \mathbb{F}$ and $\zeta \in \mu_n$. Therefore, the action of $\widetilde{G}$ on a function $\eta_f \in L^2(\mathbb{F}, \mathbb{C}^n)$ is completely determined by the action of the above matrices. These actions can be expressed explicitly as follows. Let δ, γ and x be in $\mathbb{F}^\times$.

- The action of $(\mathrm{dg}(\delta), \zeta)$: Note that

$$(\mathrm{lt}(x), 1)(\mathrm{dg}(\delta), \zeta) = (\mathrm{dg}(\delta), \zeta)(\mathrm{lt}(\delta^2 x), 1).$$

So,

$$\begin{aligned}(\mathrm{dg}(\delta), \zeta) \cdot \eta_f(x) &= f\,(\mathrm{lt}(x), 1)(\mathrm{dg}(\delta), \zeta)) \\ &= f\left((\mathrm{dg}(\delta), \zeta)(\mathrm{lt}(\delta^2 x), 1)\right) \\ &= \epsilon(\zeta)|\delta|\rho(\iota(\delta)) f\left(\mathrm{lt}(\delta^2 x), 1\right).\end{aligned}$$

Hence,

$$(\mathrm{dg}(\delta), \zeta) \cdot \eta_f(x) = \epsilon(\zeta)|\delta|\rho(\iota(\delta))\eta_f(\delta^2 x). \tag{6.1.9}$$

- The action of $(\mathrm{lt}(\gamma), 1)$: Note that

$$\begin{aligned}(\mathrm{lt}(x), 1)\,(\mathrm{lt}(\gamma), 1) &= \left(\mathrm{lt}(x+\gamma), (\frac{x+\gamma}{x}, \frac{x+\gamma}{\gamma})_n\right) \\ &= (\mathrm{lt}(x+\gamma), 1)\,, \quad \text{by Corollary 2.2.5.}\end{aligned}$$

Therefore,

$$(\mathrm{lt}(\gamma), 1) \cdot \eta_f(x) = f\,((\mathrm{lt}(x), 1)\,(\mathrm{lt}(\gamma), 1)) = f\,(\mathrm{lt}(x+\gamma), 1)\,.$$

Hence,

$$(\mathrm{lt}(\gamma), 1) \cdot \eta_f(x) = \eta_f(x+\gamma). \tag{6.1.10}$$

- The action of $\widetilde{w}$: Note that

$$(\mathrm{lt}(x), 1)\,\widetilde{w} = (\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}, 1)$$

$$= \left(\begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}, 1\right)\left(\mathrm{lt}(-x^{-1}), (-x,x)_n\right)$$
$$= \left(\begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}, 1\right)\left(\mathrm{lt}(-x^{-1}), 1\right),$$

since $(-x,x)_n = 1$. So,

$$\begin{aligned} \widetilde{w} \cdot \eta_f(x) &= f\left((\mathrm{lt}(x), 1)\,\widetilde{w}\right) \\ &= f\left(\left(\begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}, 1\right)\left(\mathrm{lt}(-x^{-1}), 1\right)\right). \end{aligned}$$

Hence,

$$\widetilde{w} \cdot \eta_f(x) = |x^{-1}|\rho(\iota(x^{-1}))\eta_f(-x^{-1}). \tag{6.1.11}$$

∎

For every $\mathfrak{h} \in L^2(\mathbb{F}, \mathbb{C}^n)$, let $\widehat{\mathfrak{h}}$ be the Fourier transform of $\mathfrak{h}$, defined in Definition 1.1.14 by

$$\widehat{\mathfrak{h}}(u) := \int_{\mathbb{F}} \lambda(-ux)\mathfrak{h}(x)dx.$$

This defines the map

$$\begin{aligned} \mathcal{F} : L^2(\mathbb{F}, \mathbb{C}^n) &\to L^2(\mathbb{F}, \mathbb{C}^n) \\ \mathfrak{h} &\mapsto \widehat{\mathfrak{h}}. \end{aligned}$$

Proposition 6.1.5. *Let δ and $\gamma \in \mathbb{F}^\times$, and $\zeta \in \mu_n$. The action of $\widetilde{G}$ on $\mathcal{F}\circ\boldsymbol{\eta}\left(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho\right)$ is completely determined by the following formulas:*

$$\begin{aligned} (\mathrm{dg}(\delta), \zeta) \cdot \widehat{\mathfrak{h}}(u) &= \epsilon(\zeta)|\delta|^{-1}\rho(\iota(\delta))\widehat{\mathfrak{h}}(u\delta^{-2}), \\ (\mathrm{lt}(\gamma), 1) \cdot \widehat{\mathfrak{h}}(u) &= \lambda(u\gamma)\widehat{\mathfrak{h}}(u), \end{aligned}$$

$$\widetilde{w}\cdot\widehat{\mathfrak{h}}(u) \quad = \quad \int_{\mathbb{F}} J_u(y)\widehat{\mathfrak{h}}(y)dy,$$

where $\widehat{\mathfrak{h}} \in \mathcal{F}\circ\boldsymbol{\eta}\left(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho\right) \subset L^2(\mathbb{F},\mathbb{C}^n)$ *and* $u \in \mathbb{F}$, *and*

$$J_u(y) = \int_{\mathbb{F}^\times} \lambda(-ux - yx^{-1})\rho(\iota(x^{-1}))\frac{dx}{|x|}.$$

Proof: We compute the action of $\widetilde{G}$ on $\mathcal{F}\circ\boldsymbol{\eta}\left(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho\right)$ induced by $\mathcal{F}$.

- The action of $(\mathrm{dg}(\delta),\zeta)$:

$$\begin{aligned}
[(\widehat{\mathrm{dg}(\delta),\zeta)\cdot}\mathfrak{h}](u) &= \int_{\mathbb{F}} \lambda(-ux)\epsilon(\zeta)|\delta|\rho(\iota(\delta))\mathfrak{h}(\delta^2x)dx \qquad (6.1.12)\\
&= \epsilon(\zeta)|\delta|\rho(\iota(\delta))\int_{\mathbb{F}}\lambda(-ux)\mathfrak{h}(\delta^2x)dx\\
&= \epsilon(\zeta)|\delta|\rho(\iota(\delta))\int_{\mathbb{F}}\lambda(-u\delta^{-2}y)\mathfrak{h}(y)|\delta|^{-2}dy, \quad (y=\delta^2x)\\
&= \epsilon(\zeta)|\delta|^{-1}\rho(\iota(\delta))\widehat{\mathfrak{h}}(u\delta^{-2}).
\end{aligned}$$

- The action of $(\mathrm{lt}(\gamma),1)$:

$$\begin{aligned}
[(\widehat{\mathrm{lt}(\gamma),1)\cdot}\mathfrak{h}](u) &= \int_{\mathbb{F}} \lambda(-ux)\mathfrak{h}(x+\gamma)dx \qquad (6.1.13)\\
&= \int_{\mathbb{F}} \lambda\left(-u(y-\gamma)\right)\mathfrak{h}(y)dy \quad (y=x+\gamma)\\
&= \lambda(u\gamma)\int_{\mathbb{F}}\lambda(-uy)\mathfrak{h}(y)dy\\
&= \lambda(u\gamma)\widehat{\mathfrak{h}}(u).
\end{aligned}$$

- The action of $\widetilde{w}$:

$$\widehat{[\widetilde{w}\cdot\mathfrak{h}]}(u) = \int_{\mathbb{F}}\lambda(-ux)\rho(\iota(x^{-1}))\mathfrak{h}(-x^{-1})\frac{dx}{|x|}$$

Applying the inverse Fourier transform, given in Definition 1.1.14, we have

$$\widehat{[\widetilde{w}\cdot\mathfrak{h}]}(u) \quad = \quad \int_{\mathbb{F}}\lambda(-ux)\rho(\iota(x^{-1}))\left(\int_{\mathbb{F}}\lambda(-yx^{-1})\widehat{\mathfrak{h}}(y)dy\right)\frac{dx}{|x|} \quad (6.1.14)$$

$$= \int_{\mathbb{F}} J_u(y)\widehat{\mathfrak{h}}(y)dy,$$

where

$$J_u(y) = \int_{\mathbb{F}} \lambda(-ux - yx^{-1})\rho(\iota(x^{-1}))\frac{dx}{|x|}.$$

∎

Remark 6.1.6. Notice the resemblance between $J_u(y)$ in Proposition 6.1.5 and the Bessel function defined in Definition 1.1.20. In fact $J_u(y)$ is a matrix whose entries can be explicitly expressed in terms of Bessel functions. We will calculate those entries in Section 6.1.3.

Indeed, Proposition 6.1.5 defines an action of $\widetilde{G}$ on $L^2(\mathbb{F}, \mathbb{C}^n)$, and $\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho) \subset L^2(\mathbb{F}, \mathbb{C}^n)$ with this action is a $\widetilde{G}$-space isomorphic to $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ via $\mathcal{F} \circ \boldsymbol{\eta}$. We will refer to the image of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ under $\mathcal{F} \circ \boldsymbol{\eta}$ together with the above action as the λ-*realization* of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$.

6.1.2 An Intertwining Operator

We say A is an intertwining operator on $\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho)$ if A is a bounded linear operator on the space $L^2(\mathbb{F}, \mathbb{C}^n)$ such that it commutes with the action of $\widetilde{G}$, given in 6.1.5; that is,

$$(\mathsf{A}(\mathsf{g} \cdot \mathfrak{h})) = (\mathsf{g} \cdot \mathsf{A}(\mathfrak{h})), \tag{6.1.15}$$

for every $\mathsf{g} \in \widetilde{G}$ and $\mathfrak{h} \in L^2(\mathbb{F}, \mathbb{C}^n)$. Since $\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho)$ is equal to the space of smooth vectors in $L^2(\mathbb{F}, \mathbb{C}^n)$, A maps $\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho)$ to itself.

It follows from Theorem 5.3.6 that $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$, and equivalently its λ-realization, decomposes into two inequivalent subrepresentations. In order to study the distribution of K-types among these subspaces, we will realize these $\widetilde{G}$-subspaces as the kernel of intertwining operators on $\mathcal{F} \circ \boldsymbol{\eta}(\operatorname{ind}_{\widetilde{B}}^{\widetilde{G}}\rho)$. So we would like to describe all the intertwining operators $\mathsf{A} \in \operatorname{End}_{\widetilde{G}}(L^2(\mathbb{F}, \mathbb{C}^n))$.

Definition 6.1.7. *Fix $k \geq 1$. Let $z : \mathbb{F} \to \mathrm{M}_k(\mathbb{C})$ be a bounded, measurable function. The* multiplication operator $\mathrm{T}_z \in \operatorname{End}\left(L^2(\mathbb{F}, \mathbb{C}^k)\right)$ *is defined by*

$$\mathrm{T}_z(\mathfrak{h})(u) = z(u)\mathfrak{h}(u),$$

for all $\mathfrak{h} \in L^2(\mathbb{F}, \mathbb{C}^k)$ and $u \in \mathbb{F}$.

For each $\gamma \in \mathbb{F}$, define

$$_k\lambda_\gamma : \mathbb{F} \to \mathrm{M}_k(\mathbb{C}), \quad x \mapsto \lambda(\gamma x)\mathrm{I}_k. \tag{6.1.16}$$

It follows from Proposition 6.1.5 that $(\mathrm{lt}(\gamma), 1)$, $\gamma \in \mathbb{F}$, acts by the multiplication operator $\mathrm{T}_{_n\lambda_\gamma}$. We state the following fact, the proof of which can be found in Proposition [Rob83, 18.1].

Proposition 6.1.8. *Let B be a bounded operator in $L^2(\mathbb{F}, \mathbb{C})$ that commutes with all multiplication operators by a bounded scalar-valued function. Then B itself is given by multiplication by a bounded function. That is $\mathsf{B} = \mathrm{T}_z$ for some $z \in L^\infty(\mathbb{F}, \mathbb{C})$.*

We use Proposition 6.1.8 to show that A is a multiplication operator, given pointwise by $n \times n$ matrices.

Lemma 6.1.9. *Let A be as in* (6.1.15)*. Then there exists a function $\mathsf{a} : \mathbb{F} \to \mathrm{M}_n(\mathbb{C})$ such that $\mathsf{A} = \mathrm{T}_{\mathsf{a}}$.*

Proof: Note that $L^2(\mathbb{F}, \mathbb{C}^n) \simeq L^2(\mathbb{F}, \mathbb{C}) \otimes \mathbb{C}^n \simeq L^2(\mathbb{F}, \mathbb{C})^{\oplus n}$. Since A is in $\operatorname{End}\left(L^2(\mathbb{F}, \mathbb{C}^n)\right) \cong \operatorname{End}\left(L^2(\mathbb{F}, \mathbb{C})^{\oplus n}\right)$, we can write $\mathsf{A} = (\mathsf{A}_{ij})$, where $\mathsf{A}_{ij} \in \operatorname{End}(L^2(\mathbb{F}, \mathbb{C}))$. It follows from (6.1.13) and (6.1.15) that A commutes with the multiplication operator $\mathrm{T}_{_n\lambda_\gamma}$, for all $\gamma \in \mathbb{F}$. Therefore, each A_{ij} commutes with $\mathrm{T}_{_1\lambda_\gamma}$, for all $\gamma \in \mathbb{F}$. Observe that the map

$$\mathbb{F} \to \hat{\mathbb{F}}, \quad \gamma \mapsto {_1\lambda_\gamma}$$

is an isomorphism. We deduce that the operator A_{ij} commutes with T_z for all $z \in \hat{\mathbb{F}}$. Hence, by Proposition 6.1.8, every A_{ij} is given point-wise by a multiplication operator $\mathrm{T}_{\mathsf{a}_{ij}}$ for some function $\mathsf{a}_{ij} \in L^\infty(\mathbb{F}, \mathbb{C})$. Let $\mathsf{a} = (\mathsf{a}_{ij})$. It follows that $\mathsf{A} = \mathrm{T}_{\mathsf{a}}$.

∎

Proposition 6.1.10. *Suppose* A *is an intertwining operator, as in* (6.1.15)*, and write* $A = T_{\mathsf{a}}$*, as in the previous lemma. Then* $\mathsf{a} : \mathbb{F} \to M_n(\mathbb{C})$*, is uniquely determined by its values on* $\mathbb{F}^\times/{\mathbb{F}^\times}^2$ *via*

$$\mathsf{a}(u\delta^2) = \rho(\iota(\delta))\mathsf{a}(u)\rho(\iota(\delta^{-1})), \tag{6.1.17}$$

for all $\delta \in \mathbb{F}^\times$ *and* $u \in \mathbb{F}$*, and satisfies*

$$\mathsf{a}(u)J_u(y) = J_u(y)\mathsf{a}(y), \tag{6.1.18}$$

for all $u, y \in \mathbb{F}$*.*

Proof: Let $\delta \in \mathbb{F}^\times$ and set $\mathsf{g} = \iota(\delta^{-1})$. By Proposition 6.1.5,

$$\left(\mathsf{A}(\mathsf{g} \cdot \widehat{\mathfrak{h}})\right)(u) = \mathsf{a}(u)(\mathsf{g} \cdot \widehat{\mathfrak{h}})(u) = \mathsf{a}(u)\rho(\iota(\delta^{-1}))|\delta|\widehat{\mathfrak{h}}(u\delta^2),$$

whereas

$$\left(\mathsf{g} \cdot \mathsf{A}(\widehat{\mathfrak{h}})\right)(u) = \rho(\iota(\delta^{-1}))|\delta|\mathsf{a}(u\delta^2)\widehat{\mathfrak{h}}(u\delta^2).$$

Since A commutes with the action of g, we conclude

$$\mathsf{a}(u\delta^2) = \rho(\iota(\delta))\mathsf{a}(u)\rho(\iota(\delta^{-1})), \tag{6.1.19}$$

for all $\delta \in \mathbb{F}^\times$ and $u \in \mathbb{F}$. Therefore, the intertwining operator A is determined completely by the values of a on the square classes of $\mathbb{F}^\times$.

Because A commutes with the action of $\widetilde{w}$, it follows from (6.1.14) that

$$\int_{\mathbb{F}} \mathsf{a}(u)J_u(y)\widehat{\mathfrak{h}}(y)dy \;=\; \int_{\mathbb{F}} J_u(y)\mathsf{a}(y)\widehat{\mathfrak{h}}(y)dy,$$

for all $\widehat{\mathfrak{h}} \in L^2(\mathbb{F}, \mathbb{C}^n)$. This implies that

$$\mathsf{a}(u)J_u(y) = J_u(y)\mathsf{a}(y), \tag{6.1.20}$$

for all $u, y \in \mathbb{F}$. ∎

In order to realise the subrepresentations of $\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho)$ as the kernel of an intertwining operator A, we need to calculate a, which is completely determined by its values on $\mathbb{F}^\times/\mathbb{F}^{\times 2}$. We start by calculating $\mathsf{a}(1)$. We will see that $\mathsf{a}(1)$ gives us enough information to determine the distribution of some of the first level K-types.

6.1.3 Calculating the Matrix $\mathsf{a}(1)$

The next lemma shows that, to calculate $\mathsf{a}(1)$, we first need to calculate $J_1(y)$.

Lemma 6.1.11. *The matrix $\mathsf{a}(1)$ commutes with the matrices $J_1(\delta^2)\rho(\iota(\delta))$ for any $\delta \in \mathcal{O}^\times$.*

Proof: Let $u = 1$ and $y = \delta^2$ in (6.1.17) and (6.1.18) in Proposition 6.1.10. Note that $\delta \in \mathcal{O}^\times$ implies that $\rho(\iota(\delta^{-1})) = \rho(\iota(\delta))^{-1}$. Then,

$$\begin{aligned} \mathsf{a}(1)J_1(\delta^2)\rho(\iota(\delta)) &= J_1(\delta^2)\mathsf{a}(\delta^2)\rho(\iota(\delta)) \\ &= J_1(\delta^2)\rho(\iota(\delta))\mathsf{a}(1)\rho(\iota(\delta^{-1}))\rho(\iota(\delta)) \\ &= J_1(\delta^2)\rho(\iota(\delta))\mathsf{a}(1). \end{aligned}$$

∎

Recall that

$$J_u(y) = \int_{\mathbb{F}} \lambda(-ux - yx^{-1})\rho(\iota(x^{-1}))\frac{dx}{|x|}. \tag{6.1.21}$$

Set $L := \{x \in \mathbb{F} \mid n|\mathrm{val}(x)\}$, so that $\varpi^k L$ is the set of all elements of valuation k mod n. Then

$$\mathbb{F}^\times = \bigcup_{k=0}^{n-1} \varpi^k L.$$

We will calculate the integral in (6.1.21) over each $\varpi^k L$, $0 \leq k < n$. Let $\rho_{i,j}$ denote the (i,j)-th matrix coefficient of ρ.

Lemma 6.1.12. *Let $x \in \mathbb{F}^\times$, then*

$$\rho_{i,j}(\iota(x^{-1})) = \begin{cases} (-1)^{\mathrm{val}(x)+i-j}\epsilon\left(\vartheta(x)^{-2+i+j}\right), & \text{if } i-j \equiv \mathrm{val}(x) \mod n \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.22}$$

Proof: Fix k. If $x \in \varpi^k L$, then $\varpi^k x^{-1} \in L$. So,

$$\rho(\iota(x^{-1})) = \rho(\iota(\varpi^{-k}))\rho\left(\mathrm{dg}(\varpi^k x^{-1}), (x,\varpi)_n^{-k}\right).$$

It follows from (6.1.3) in Lemma 6.1.1 that

$$\begin{aligned} &\rho\left(\mathrm{dg}(\varpi^k x^{-1}), (x,\varpi)_n^{-k}\right) \\ = \quad &(-1)^{\mathrm{val}(x^{-1})+k} \begin{pmatrix} \epsilon\left(\vartheta(x)^k\right) & 0 & \dots & 0 \\ 0 & \epsilon\left(\vartheta(x)^{k+2}\right) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \epsilon\left(\vartheta(x)^{k+2(n-1)}\right) \end{pmatrix}. \end{aligned}$$

The result follows from (6.1.4) and matrix multiplication. ∎

Let ε be a fixed element of order n in $\mathcal{O}^\times/\mathcal{O}^{\times n}$. Observe that $(\varepsilon,\varpi)_n$ is a primitive n-th root of unity. Without loss of generality, we can choose ε such that $\epsilon\left((\varepsilon,\varpi)_n\right) = e^{\frac{2\pi i}{n}}$. Set $\xi = \epsilon\left((\varepsilon,\varpi)_n\right) = e^{\frac{2\pi i}{n}}$. Define,

$$\Lambda_k(x) := \frac{1}{n}\sum_{i=0}^{n-1} \xi^{i(\mathrm{val}(x)-k)}\rho(\iota(x^{-1})). \tag{6.1.23}$$

Since

$$\sum_{i=0}^{n-1} \xi^{il} = \begin{cases} n, & \text{if } l \equiv 0 \mod n \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\Lambda_k(x) = \begin{cases} \rho(\iota(x^{-1})), & \text{if } \mathrm{val}(x) \equiv k \mod n \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.24}$$

Therefore,

$$\begin{aligned} J_u(y) &= \int_{\mathbb{F}} \lambda(-ux - yx^{-1})\rho(\iota(x^{-1}))\frac{dx}{|x|} \\ &= \sum_{k=0}^{n-1} \int_{\varpi^k L} \lambda(-ux - yx^{-1})\rho(\iota(x^{-1}))\frac{dx}{|x|}, \end{aligned} \tag{6.1.25}$$

which by property (6.1.24) of Λ_k equals

$$\sum_{k=0}^{n-1} \int_{\mathbb{F}} \lambda(-ux - yx^{-1})\Lambda_k(x)\frac{dx}{|x|}.$$

If follows from (6.1.25), (6.1.23) and Lemma 6.1.12 that the (i,j)-th entry of $J_u(y)$ is

$$\frac{1}{n}\int_{\mathbb{F}} \sum_{k=0}^{n-1} \xi^{k(\mathrm{val}(x)-i+j)}(-1)^{\mathrm{val}(x)+i-j}\epsilon\left(\vartheta(x)^{-2+i+j}\right)\lambda(-ux - yx^{-1})\frac{dx}{|x|}. \tag{6.1.26}$$

For each $0 \leq i, j, k \leq n$, define

$$\Theta_{k,i,j}(x) := \xi^{k\mathrm{val}(x)}(-1)^{\mathrm{val}(x)}\epsilon\left(\vartheta(x)^{-2+i+j}\right). \tag{6.1.27}$$

Lemma 6.1.13. *The character $\Theta_{k,i,j}$ has the following properties.*

1. *Let $p_k := (1 - \frac{2k}{n})\mathbf{s}$. Then*

$$\Theta_{k,i,j}(x) = |x|^{p_k}\epsilon\left(\vartheta(x)^{-2+i+j}\right). \tag{6.1.28}$$

2. *For all $x \in \mathbb{F}^\times$, $\Theta_{k,i,j}(-x) = \Theta_{k,i,j}(x)$.*

3. *If $2 - i - j \equiv 0 \mod n$, then $\Theta_{k,i,j}$ is unramified; otherwise $\Theta_{k,i,j}$ is ramified of ramification degree one.*

Proof: Note that $\xi^{\mathrm{val}(x)} = |x|^{\frac{-2\pi i}{n\log q}}$ and $(-1)^{\mathrm{val}(x)} = |x|^{\frac{\pi i}{\log q}}$. Hence,

$$\Theta_{k,i,j}(x) = |x|^{p_k}\epsilon\left(\vartheta(x)^{-2+i+j}\right), \tag{6.1.29}$$

where

$$\begin{aligned} p_k : &= k\frac{-2\pi i}{n\log q} + \frac{\pi i}{\log q} \\ &= \frac{\pi i(n-2k)}{n\log q} = (1-\frac{2k}{n})\mathbf{s}. \end{aligned}$$

Part (2) follows from observing that, because n is odd, $(-1,\varpi)_n = 1$; hence, $(-x,\varpi)_n = (x,\varpi)_n$ for all $x\in\mathbb{F}$. Finally, note that $-2+i+j\equiv 0 \mod n$ implies that $\Theta_{k,i,j} = |x|^{p_k}$, and hence is unramified. Otherwise, using Definition 2.2.2, it is easy to see that $\Theta_{k,i,j}|_{1+\mathfrak{p}} = 1$ and $\Theta_{k,i,j}|_{\mathcal{O}} \neq 1$, and hence the character is ramified of degree one. ∎

Using Lemma 6.1.13, the (i,j)-th entry of $J_u(y)$ in (6.1.26) can be written as

$$\frac{(-1)^{i-j}}{n}\left[\sum_{k=0}^{n-1}\xi^{k(-i+j)}\left(\int_{\mathbb{F}}\Theta_{k,i,j}(x)\lambda(-ux-yx^{-1})\frac{dx}{|x|}\right)\right]. \tag{6.1.30}$$

Note that the integral in (6.1.30) is a Bessel function $J_{\Theta_{k,i,j}}(-u,-y)$ defined in Definition 1.1.20, and can be expressed in terms of the Gamma function, by Theorem 1.1.21. Set $\alpha = 2-i-j$. Then by Lemma 6.1.13, $\Theta_{k,i,j}(x) = |x|^{p_k}\epsilon\left(\vartheta(x)^{-\alpha}\right)$. It follows from Theorem 1.1.19 that if $\Theta_{k,i,j}$ is ramified, whence by Lemma 6.1.13 ramified of degree one, then there exists a constant in $\mathbb{C}$ depending on $\epsilon\circ\vartheta^{-\alpha}$, which we denote by C_α, such that $|C_\alpha| = 1$ and $C_\alpha C_{-\alpha} = 1$, and that $\Gamma(\Theta_{k,i,j}) = C_\alpha q^{p_k-\frac{1}{2}}$, where $p_k = (1-\frac{2k}{n})\mathbf{s}$. The next proposition gives the values of $J_{\Theta_{k,i,j}}(-u,-v)$ in terms of C_α and $C_{-\alpha}$, when $u,y\in\mathcal{O}^\times$.

Proposition 6.1.14. *Let $u,y\in\mathcal{O}^\times$, and let $\alpha = 2-i-j$. Then $J_{\Theta_{k,i,j}}(-u,-v) = \int_{\mathbb{F}}\Theta_{k,i,j}(x)\lambda(-ux-yx^{-1})\frac{dx}{|x|}$ is equal to*

- $-q^{-\frac{1}{2}}\left[\epsilon\left(\vartheta(y)\right)^{-\alpha}C_{-\alpha}\xi^k + \epsilon\left(\vartheta(u)\right)^{\alpha}C_\alpha\xi^{-k}\right]$, *if* $\alpha\not\equiv 0 \mod n$

- $1 + q^{-1}\xi^{-k}\frac{\xi^{3k}+1}{1+\xi^k}$, *if* $\alpha \equiv 0 \mod n$.

Proof: Set $\Theta := \Theta_{k,i,j}$. It follows from Theorem 1.1.21 and Lemma 6.1.13 that for $u, y \in \mathcal{O}^\times$,

$$\int_{\mathbb{F}} \Theta(x)\lambda(-ux - yx^{-1})\frac{dx}{|x|} = \Theta(y)\Gamma(\Theta^{-1}) + \Theta^{-1}(u)\Gamma(\Theta). \tag{6.1.31}$$

By (6.1.29) in Lemma 6.1.13, $\Theta(x) = |x|^{p_k}\epsilon(\vartheta(x)^{-\alpha})$, where $p_k = (1 - \frac{2k}{n})\mathbf{s}$. Then, Theorem 1.1.19 implies that there exists a C_α satisfying $|C_\alpha| = 1$ and $C_\alpha C_{-\alpha} = 1$ such that

$$\Gamma(\Theta) = \begin{cases} C_\alpha q^{p_k - \frac{1}{2}}, & \alpha \not\equiv 0 \\ \frac{1-q^{p_k-1}}{1-q^{-p_k}}, & \alpha \equiv 0. \end{cases} \tag{6.1.32}$$

The result is obtained by substituting (6.1.32) in (6.1.31), and observing that $q^{-p_k} = -\xi^k$. ∎

Proposition 6.1.15. *If $u, y \in \mathcal{O}^\times$ then the (i,j)-th entry of $J_u(y)$ is*

$$\begin{cases} (-1)^{i-j+1}q^{-\frac{1}{2}}C_\alpha\epsilon\left(\vartheta(u)^\alpha\right), & \text{if } -i+j \equiv 1 \mod n \text{ and } i \neq \frac{n+1}{2} \\ (-1)^{i-j+1}q^{-\frac{1}{2}}C_\alpha^{-1}\epsilon\left(\vartheta(y)^{-\alpha}\right), & \text{if } -i+j \equiv -1 \mod n \text{ and } i \neq \frac{n+3}{2} \\ 1 - \frac{1}{q}, & \text{if } i = j = 1 \\ -\frac{1}{q}, & \text{if } \{i,j\} = \{\frac{n+1}{2}, \frac{n+3}{2}\} \\ 0, & \text{otherwise,} \end{cases} \tag{6.1.33}$$

where $\alpha = 2 - i - j$.

Proof: First assume $\alpha \not\equiv 0 \mod n$. By (6.1.30) and Proposition 6.1.14, the (i,j)-th entry of $J_u(y)$ is

$$\frac{(-1)^{i-j+1}}{n}\left[q^{-\frac{1}{2}}\epsilon\left((y,\varpi)_n^\alpha\right)C_\alpha^{-1}\sum_{k=0}^{n-1}\xi^{k(-i+j+1)} + q^{-\frac{1}{2}}\epsilon\left((u,\varpi)_n^{-\alpha}\right)C_\alpha\sum_{k=0}^{n-1}\xi^{k(-i+j-1)}\right].$$

The first summand of the sum above vanishes if $n \nmid -i+j+1$ and is equal to $nq^{-\frac{1}{2}}\epsilon\left((y,\varpi)_n^{\alpha}\right)C_{\alpha}^{-1}$ otherwise. The second summand vanishes if $n \nmid -i+j-1$ and is equal to $nq^{-\frac{1}{2}}\epsilon\left((u,\varpi)_n^{-\alpha}\right)C_{\alpha}$ otherwise.

Now suppose $\alpha \equiv 0 \mod n$. By (6.1.30) and Proposition 6.1.14, the (i,j)-th entry of $J_u(y)$ is

$$\frac{(-1)^{i-j}}{n}\left[\sum_{k=0}^{n-1}\xi^{k(-i+j)}\left(1+q^{-1}\xi^{-k}\frac{\xi^{3k}+1}{1+\xi^k}\right)\right]. \qquad (6.1.34)$$

Observe that $\frac{\xi^{3k}+1}{1+\xi^k} = \frac{(1+\xi^k)(1-\xi^k+\xi^{2k})}{1+\xi^k} = 1-\xi^k+\xi^{2k}$. Hence, (6.1.34) simplifies to

$$\frac{(-1)^{i-j}}{n}\left[\sum_{k=0}^{n-1}\xi^{k(-i+j)} + q^{-1}\sum_{k=0}^{n-1}\xi^{k(-i+j-1)} - q^{-1}\sum_{k=0}^{n-1}\xi^{k(-i+j)} + q^{-1}\sum_{k=0}^{n-1}\xi^{k(-i+j+1)}\right].$$

Because $\alpha \equiv 0 \mod n$, either $i=j=1$ or $2 \le i \le n$ and $j = 2-i+n$. So, under the hypothesis $\alpha \equiv 0$, we have

$$\sum_{k=0}^{n-1}\xi^{k(-i+j)} = \begin{cases} n, & i=j=1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{k=0}^{n-1}\xi^{k(-i+j+1)} = \begin{cases} n, & i=\frac{n+3}{2},\ j=\frac{n+1}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sum_{k=0}^{n-1}\xi^{k(-i+j)} = \begin{cases} n, & i=\frac{n+1}{2},\ j=\frac{n+3}{2} \\ 0, & \text{otherwise.} \end{cases}$$

If follows that the (i,j)-th entry of $J_u(y)$ is equal to $1-\frac{1}{q}$ for $i=j=1$, and to $-\frac{1}{q}$ for $i=\frac{n+3}{2}$, $j=\frac{n+1}{2}$ or $i=\frac{n+1}{2}$, $j=\frac{n+3}{2}$ and zero otherwise. ∎

Example 6.1.16. *Let us spell out the case for $n=3$ explicitly. We will use the formulas* (6.1.30) *and Proposition 6.1.14 directly to calculate the entries of $J_u(y)$ for*

$u, y \in \mathcal{O}^\times$. Note that, for $n = 3$, we have $p_k = (1 - \frac{2k}{3})\mathbf{s}$. We will frequently use the following equalities, which are easy to verify,

$$q^{p_0} = q^{-p_0} = -1, \quad q^{-p_1} = q^{p_2} = -\xi, \quad q^{p_1} = q^{-p_2} = -\xi^2.$$

- *$(1,1)$-entry: Note that in this case $\alpha = 2 - 1 - 1 = 0$. Hence the $(1,1)$-th entry of $J_u(y)$ is equal to*

$$\begin{aligned} & \frac{1}{3}\sum_{k=0}^{2}\left[\int_{\mathbb{F}} |x|^{p_k}\lambda(-ux - yx^{-1})\frac{dx}{|x|}\right] \\ = \ & \frac{1}{3}\sum_{k=0}^{2}\left[1 + q^{-1}\xi^{-k}\frac{\xi^{3k}+1}{1+\xi^k}\right] \\ = \ & \frac{1}{3}\sum_{k=0}^{2}\left[1 + q^{-1}(\xi^{-k} - 1 + \xi^k)\right] \\ = \ & 1 - \frac{1}{q}. \end{aligned}$$

- *$(2,2)$-entry: Note that $\alpha \equiv 1 \mod 3$, Hence the $(2,2)$-th entry is*

$$\begin{aligned} & \frac{1}{3}\sum_{k=0}^{2}\left[-q^{-\frac{1}{2}}\left[\epsilon\left((y,\varpi)_3\right)C_2\xi^k + \epsilon\left((u,\varpi)_3^2\right)C_1\xi^{-k}\right]\right] \\ = \ & -q^{-\frac{1}{2}}(1+\xi+\xi^2)\left(\epsilon\left((y,\varpi)_3\right)C_2\right) - (1+\xi^2+\xi)\left(\epsilon\left((u,\varpi)_3^2\right)C_2^{-1}\right) \\ = \ & 0. \end{aligned}$$

  Similarly, the $(3,3)$-entry is zero.

- *$(1,2)$-entry: Here, $\alpha \equiv 2 \mod 3$ and $i - j = -1$. Hence, the $(1,2)$-the entry is*

$$\begin{aligned} & -\frac{1}{3}\sum_{k=0}^{2}\xi^k\left[-q^{-\frac{1}{2}}\left[\epsilon\left((y,\varpi)_3^2\right)C_2^{-1}\xi^k + \epsilon\left((u,\varpi)_3\right)C_2\xi^{-k}\right]\right] \\ = \ & -\frac{1}{3}\left[(-1-\xi^2-\xi^4)\epsilon\left({(y,\varpi)_3}^2\right)C_2^{-1}q^{-\frac{1}{2}} + (-1-1-1)\epsilon\left((u,\varpi)_3\right)C_2q^{-\frac{1}{2}}\right] \\ = \ & \epsilon\left((u,\varpi)_3\right)C_2q^{-\frac{1}{2}}. \end{aligned}$$

- $(1,3)$-*entry: Here* $\alpha \equiv 1 \mod 3$ *and* $i-j=-2$. *Hence, the* $(1,3)$-*th entry is*

$$\begin{aligned}
&\frac{1}{3}\sum_{k=0}^{2}\xi^{2k}\left[-q^{-\frac{1}{2}}\left[\epsilon\left((y,\varpi)_3\right)C_2\xi^k+\epsilon\left((u,\varpi)_3^2\right)C_2^{-1}\xi^{-k}\right]\right]\\
=\ &\frac{1}{3}\left[\epsilon\left((y,\varpi)_3\right)C_2q^{-\frac{1}{2}}(-1-\xi^3-\xi^6)+\epsilon\left((u,\varpi)_3^2\right)C_2^{-1}q^{-\frac{1}{2}}(-1-\xi-\xi^5)\right]\\
=\ &-(y,\varpi)_3C_2q^{-\frac{1}{2}}.
\end{aligned}$$

- $(2,1)$-*entry: Here* $\alpha \equiv 2 \mod 3$ *and* $i-j=1$. *Hence, the* $(2,1)$-*th entry is*

$$\begin{aligned}
&-\frac{1}{3}\sum_{k=0}^{2}\xi^{-k}\left[-q^{-\frac{1}{2}}\left[\epsilon\left((y,\varpi)_3^2\right)C_2^{-1}\xi^k+\epsilon\left((u,\varpi)_3\right)C_2\xi^{-k}\right]\right]\\
=\ &-\frac{1}{3}\left[\epsilon\left((y,\varpi)_3^2\right)C_2^{-1}q^{-\frac{1}{2}}(-1-1-1)+\epsilon\left((u,\varpi)_3\right)C_2q^{-\frac{1}{2}}(-1-\xi-\xi^{-1})\right]\\
=\ &\epsilon\left((y,\varpi)_3^2\right)C_2^{-1}q^{-\frac{1}{2}}.
\end{aligned}$$

- $(2,3)$-*entry: Here* $\alpha \equiv 0 \mod 3$ *and* $i-j=-1$. *Hence, the* $(2,3)$-*th entry is*

$$\begin{aligned}
&-\frac{1}{3}\sum_{k=0}^{2}\xi^{k}\left[1+q^{-1}\xi^{-k}\frac{\xi^{3k}+1}{1+\xi^k}\right]\\
=\ &-\frac{1}{3}\sum_{k=0}^{2}\left[\xi^k+q^{-1}-\xi^k+\xi^{2k}\right]\\
=\ &-\frac{1}{q}.
\end{aligned}$$

- $(3,1)$-*entry: Here* $\alpha \equiv 1 \mod 3$ *and* $i-j=2$. *Hence, the* $(2,1)$-*th entry is*

$$\begin{aligned}
&\frac{1}{3}\sum_{k=0}^{2}\xi^{k}\left[-q^{-\frac{1}{2}}\left[\epsilon\left((y,\varpi)_3\right)C_2\xi^k+\epsilon\left((u,\varpi)_3^2\right)C_2^{-1}\xi^{-k}\right]\right]\\
=\ &\frac{1}{3}\left[\epsilon\left((y,\varpi)_3\right)C_2q^{-\frac{1}{2}}(-1-\xi^2-\xi^4)+\epsilon\left((u,\varpi)_3^2\right)C_2^{-1}q^{-\frac{1}{2}}(-1-\xi^3-\xi^3)\right]\\
=\ &-\epsilon\left((u,\varpi)_3^2\right)C_2^{-1}q^{-\frac{1}{2}}.
\end{aligned}$$

- $(3,2)$-*entry: Here* $\alpha \equiv 0 \mod 3$ *and* $i-j=1$. *Hence, the* $(3,2)$-*th entry is*

$$\begin{aligned} & -\frac{1}{3}\sum_{k=0}^{2}\xi^{-k}\left[1+q^{-1}\xi^{-k}\frac{\xi^{3k}+1}{1+\xi^{k}}\right] \\ = & -\frac{1}{3}\sum_{k=0}^{2}\left(\xi^{-k}+q^{-1}(\xi^{-2k}-\xi^{k}+1)\right) \\ = & -\frac{1}{q}. \end{aligned}$$

Hence, for $u, y \in \mathcal{O}^\times$,

$$J_u(y) = \begin{pmatrix} 1-\frac{1}{q} & \epsilon\left(\vartheta(u)^2\right)C_2 q^{-\frac{1}{2}} & -\epsilon\left(\vartheta(y)^2\right)C_2 q^{-\frac{1}{2}} \\ \epsilon\left(\vartheta(y)\right)C_2^{-1}q^{-\frac{1}{2}} & 0 & -\frac{1}{q} \\ -\epsilon\left(\vartheta(u)\right)C_2^{-1}q^{-\frac{1}{2}} & \frac{1}{q} & 0 \end{pmatrix}.$$

Example 6.1.17. *Similarly one can see that for $n=5$, $J_u(y)$ is equal to*

$$\begin{pmatrix} 1-\frac{1}{q} & \epsilon\left(\vartheta(u)^4\right)C_4 q^{-\frac{1}{2}} & 0 & 0 & -\epsilon\left(\vartheta(y)^4\right)C_1^{-1}q^{-\frac{1}{2}} \\ \epsilon\left(\vartheta(y)\right)C_4^{-1}q^{-\frac{1}{2}} & 0 & \epsilon\left(\vartheta(u)^3\right)C_2 q^{-\frac{1}{2}} & 0 & 0 \\ 0 & \epsilon\left(\vartheta(y)^3\right)C_2^{-1}q^{-\frac{1}{2}} & 0 & -q^{-1} & 0 \\ 0 & 0 & -q^{-1} & 0 & \epsilon\left(\vartheta(u)^3\right)C_3 q^{-\frac{1}{2}} \\ -\epsilon\left(\vartheta(u)\right)C_1 q^{-\frac{1}{2}} & 0 & 0 & \epsilon\left(\vartheta(y)^2\right)C_3^{-1}q^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Next, we calculate $\mathsf{a}(1)$. Recall that by Lemma 6.1.11, $\mathsf{a}(1)$ commutes with the matrices $J_1(\delta^2)\rho(\iota(\delta))$ for any $\delta \in \mathcal{O}^\times$. In particular, it commutes with linear combinations of the matrices $M(k) := J_1(\varepsilon^{2k})\rho(\iota(\varepsilon^k))$, $1 \le k \le n$. Define

$$\Omega(m) := \sum_{k=1}^{n}\xi^{m(k-1)}M(k), \tag{6.1.35}$$

for $0 \le m < n$. In the next lemma, we will calculate the $\Omega(m)$, which will be used in the calculation of $\mathsf{a}(1)$. Let $\Omega_{i,j}(m)$ denote the (i,j)-th entry of $\Omega(m)$.

Lemma 6.1.18. *Let $\Omega(m)$ be as defined in* (6.1.35). *Then*

$$\Omega_{i,j}(0) = \begin{cases} n(q^{\frac{1}{2}}-q^{-\frac{1}{2}}), & i=j=1 \\ -C_1^{-1}n, & i=1, j=n \\ -C_1 n, & i=n, j=1 \\ 0, & \textit{otherwise,} \end{cases} \tag{6.1.36}$$

$$\Omega_{i,j}(1) = \begin{cases} nC_{2-n}\xi^{-1}, & i = \frac{n-1}{2}, j = \frac{n+1}{2} \\ -nq^{-\frac{1}{2}}\xi^{-1}, & i = \frac{n+3}{2}, j = \frac{n+1}{2} \\ 0, & otherwise, \end{cases} \quad (6.1.37)$$

$$\Omega_{i,j}(n-1) = \begin{cases} n\xi C_{2-n}^{-1}, & i = \frac{n+1}{2}, j = \frac{n-1}{2} \\ -n\xi q^{-\frac{1}{2}}, & i = \frac{n+1}{2}, j = \frac{n+3}{2} \\ 0, & otherwise, \end{cases} \quad (6.1.38)$$

and for $1 \leq l \leq \frac{n-3}{2}$,

$$\Omega_{i,j}(2l) = \begin{cases} nC_{2-2l-1}^{-1}\xi^{-2l}, & i = l+1, j = l \\ nC_{2+2l-1}\xi^{-2l}, & i = n-l, j = n-l+1 \\ 0, & otherwise, \end{cases} \quad (6.1.39)$$

and

$$\Omega_{i,j}(n-2l) = \begin{cases} nC_{2-2l-1}\xi^{2l}, & i = l, j = l+1 \\ nC_{2+2l-1}^{-1}\xi^{2l}, & i = n-l+1, j = n-l \\ 0, & otherwise, \end{cases} \quad (6.1.40)$$

and the above matrices exhaust $\{\Omega(m) \mid 0 \leq m < n\}$.

Proof: Using Proposition 6.1.15 and (6.1.3), it is not difficult to see that

$$M_{i,j}(k) = \begin{cases} q^{\frac{1}{2}} - q^{-\frac{1}{2}}, & i = j = 1 \\ (-1)^{i-j+1}C_{\alpha}\xi^{2ik}, & -i+j \equiv 1 \mod n, i \neq \frac{n+1}{2} \\ (-1)^{i-j+1}C_{\alpha}^{-1}\xi^{-2jk}, & -i+j \equiv -1 \mod n, i \neq \frac{n+3}{2} \\ -q^{-\frac{1}{2}}\xi^{k}, & i = \frac{n+1}{2}, j = \frac{n+3}{2} \\ -q^{-\frac{1}{2}}\xi^{-k}, & i = \frac{n+3}{2}, j = \frac{n+1}{2} \\ 0, & otherwise, \end{cases} \quad (6.1.41)$$

where $\alpha = 2 - i - j$. If $i = j = 1$ then

$$\Omega_{1,1}(m) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\xi^{-m}\sum_{k=1}^{n}\xi^{mk}$$

$$= \begin{cases} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})n\xi^{-m}, & m = 0 \\ 0, & otherwise. \end{cases}$$

Assume that $-i + j \equiv 1 \mod n$, and $i \neq \frac{n+1}{2}$. It follows from (6.1.41) that

$$\begin{aligned} \Omega_{i,j}(m) &= \sum_{k=1}^{n} \xi^{m(k-1)} M_{i,j}(k) \\ &= (-1)^{i-j+1} C_\alpha \xi^{-m} \sum_{k=1}^{n} \xi^{(2i+m)k} \\ &= \begin{cases} (-1)^{i-j+1} n C_\alpha \xi^{-m}, & n|2i+m \\ 0, & otherwise. \end{cases} \end{aligned}$$

Now assume that $-i + j \equiv -1 \mod n$ and $i \neq \frac{n+3}{2}$. Similarly, if follows that

$$\begin{aligned} \Omega_{i,j}(m) &= (-1)^{i-j+1} C_\alpha^{-1} \xi^{-m} \sum_{k=1}^{n} \xi^{(-2j+m)k} \\ &= \begin{cases} (-1)^{i-j+1} n C_\alpha^{-1} \xi^{-m}, & n|-2j+m \\ 0, & otherwise. \end{cases} \end{aligned}$$

For $i = \frac{n+1}{2}$ and $j = \frac{n+3}{2}$, we get

$$\begin{aligned} \Omega_{i,j}(m) &= -q^{-\frac{1}{2}} \xi^{-m} \sum_{k=1}^{n} \xi^{(m+1)k} \\ &= \begin{cases} -q^{-\frac{1}{2}} n \xi^{-m}, & n|m+1 \\ 0, & otherwise, \end{cases} \end{aligned}$$

and for $i = \frac{n+3}{2}$ and $j = \frac{n+1}{2}$

$$\begin{aligned} \Omega_{i,j}(m) &= -q^{-\frac{1}{2}} \xi^{-m} \sum_{k=1}^{n} \xi^{(m-1)k} \\ &= \begin{cases} -q^{-\frac{1}{2}} n \xi^{-m}, & n|m-1 \\ 0, & otherwise. \end{cases} \end{aligned}$$

The final result is a direct implication of the above calculations. ∎

Let B be an $n \times n$ matrix with entries given below.

$$B_{i,j} = \begin{cases} q^{-\frac{1}{2}}, & i = j \text{ and } i < n-j+1 \\ q^{\frac{1}{2}}, & i = j \text{ and } i > n-j+1 \\ C_{2i-1}^{-1}, & i = n-j+1 \text{ and } i < j \\ C_{2(n-i)+1}, & i = n-j+1 \text{ and } i > j \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.42}$$

Theorem 6.1.19. *Write* $\mathsf{A} = T_{\mathsf{a}}$ *as in Lemma 6.1.9. Then there exist* $\alpha, \beta \in \mathbb{C}$ *such that*

$$\mathsf{a}(1) = \alpha \mathrm{I}_n + \beta B,$$

where B *is defined in* (6.1.42).

Proof: We start by showing that all, except for the diagonal and anti-diagonal, entries of $\mathsf{a}(1)$ are zero. Lemma 6.1.11 implies that $\mathsf{a}(1)$ commutes with $M(k)$ for $1 \leq k \leq n$, and therefore it commutes with the algebra generated by these matrices, including $\Omega(m)$ for $0 \leq m < n$, which are calculated in Lemma 6.1.18.

The conditions $\mathsf{a}(1)\Omega(m) = \Omega(m)\mathsf{a}(1)$ gives a system of linear equations for the entries of $\mathsf{a}(1)$. Let $a_{i,j}$ be the (i,j)-th entry of $\mathsf{a}(1)$. First, assume that $i \neq j$ and $i \neq n-j+1$. We are in one of the following situations:

1. $i = 1$, $1 < j < n$: Consider the (n,j)-th entry of $\mathsf{a}(1)\Omega(0)$. We see that $(\Omega(0)\mathsf{a}(1))_{n,j} = \Omega_{n,1}(0)a_{1,j}$ whereas, since the j-th column of $\Omega(0)$ is zero, $(\mathsf{a}(1)\Omega(0))_{n,j} = 0$. So, since these matrices commute, $a_{1,j} = 0$.

2. $j = 1$, $1 < i < n$: By a similar argument, computing the (i,n)-th entry of $\mathsf{a}(1)\Omega(0) = \Omega(0)\mathsf{a}(1)$ yields $a_{i,1} = 0$.

3. $j = n$, $1 < i < n$: Then the $(i,1)$-th entry of $\Omega(0)\mathsf{a}(1)$ is zero. So, the $(i,1)$-th entry of $\mathsf{a}(1)\Omega(0)$ is

$$\sum_{k=1}^{n} a_{i,k}\Omega_{k,1}(0) = a_{i,1}\Omega_{1,1}(0) + a_{i,n}\Omega_{n,1}(0) = 0,$$

which implies $a_{i,n} = 0$.

4. $i = n$, $1 < j < n$: By a similar argument, computing the $(1, j)$-th entry of $\mathsf{a}(1)\Omega(0)$ yields $a_{n,j} = 0$.

5. $2 \leq j < \frac{n+1}{2}$, $2 \leq i \leq n-1$, $i \neq j$ and $i \neq n-j+1$: Then the $(i, j-1)$-entry of $\Omega\left(2(j-1)\right)\mathsf{a}(1)$ is zero. Computing the $(i, j-1)$-th entry of $\mathsf{a}(1)\Omega\left(2(j-1)\right)$ yields $a_{i,j} = 0$.

6. $\frac{n+1}{2} < j \leq n-1$ and $2 \leq i \leq n-1$, $i \neq j$ and $i \neq n-j+1$: By a similar argument, computing the $(i, j+1)$-entry of $\Omega\left(2(n-j)\right)\mathsf{a}(1) = \mathsf{a}(1)\Omega\left(2(n-j)\right)$ yields $a_{i,j} = 0$.

7. $j = \frac{n+1}{2}$ and $2 \leq i \leq n-1$ and $i \neq \frac{n+1}{2}$: Then the $(i, \frac{n-1}{2})$-th entry of $\Omega(n-1)\mathsf{a}(1)$ is zero. Computing the $(i, \frac{n-1}{2})$-th entry of $\mathsf{a}(1)\Omega(n-1)$ gives $a_{i,\frac{n+1}{2}} = 0$.

Hence, the only non-zero entries are those $a_{i,j}$'s satisfying $i = j$ or $i = n-j+1$. Commuting with $\Omega(2l)$ for $1 \leq l \leq \frac{n-3}{2}$ implies

$$a_{l,l} = a_{l+1,l+1}, \quad a_{n-l+1,n-l+1} = a_{n-l,n-l},$$

and

$$a_{n-l,l+1} = a_{n-l+1,l}C_{2l+1}C_{2l-1}^{-1}, \quad a_{l+1,n-l} = a_{l,n-l+1}C_{2l-1}C_{2l+1}^{-1}. \tag{6.1.43}$$

The equation (6.1.43) implies that for $2 \leq i \leq \frac{n-1}{2}$

$$a_{i,n-i+1} = a_{1,n}C_1C_{2i-1}^{-1},$$

and for $\frac{n+3}{2} \leq i \leq n-1$,

$$a_{i,n-i+1} = a_{n,1}C_1^{-1}C_{2(n-i)+1}.$$

Moreover, commuting with $\Omega(0)$ implies

$$a_{1,n} = a_{n,1} C_1^{-2},$$

and commuting with $\Omega(n-1)$ implies

$$a_{1,1} = a_{\frac{n+1}{2},\frac{n+1}{2}} + C_1^{-1} q^{-\frac{1}{2}} a_{n,1}, \quad a_{n,n} = a_{\frac{n+1}{2},\frac{n+1}{2}} + C_1^{-1} q^{\frac{1}{2}} a_{n,1}.$$

Let $\alpha = a_{\frac{n+1}{2},\frac{n+1}{2}}$ and $\beta = a_{n,1} C_1^{-1}$. The result follows from the above relations. ∎

Let $\{e_1, \cdots, e_n\}$ denote the standard basis for $\mathbb{C}^n$.

Lemma 6.1.20. *The eigenspaces of the matrix B defined in* (6.1.42) *are*

$$E_1 = \text{Span}\left[\left\{-e_i C_{2i-1}^{-1} q^{\frac{1}{2}} + e_{n-i+1} \mid 1 \leq i < \frac{n+1}{2}\right\} \cup \left\{e_{\frac{n+1}{2}}\right\}\right]$$

and

$$E_2 = \text{Span}\left\{e_i C_{2i-1}^{-1} q^{-\frac{1}{2}} + e_{n-i+1} \mid 1 \leq i < \frac{n+1}{2}\right\},$$

with eigenvalues 0 *and* $q^{-\frac{1}{2}} + q^{\frac{1}{2}}$ *respectively.*

Proof: It is easy to see that the eigenvalues of B are 0 and $q^{-\frac{1}{2}}+q^{\frac{1}{2}}$ of multiplicities $\frac{n+1}{2}$ and $\frac{n-1}{2}$ respectively, and after a simple calculation, one can see that E_1 is the eigenspace for the eigenvalue 0 and E_2 is the eigenspace for the eigenvalue $q^{-\frac{1}{2}} + q^{\frac{1}{2}}$. ∎

Recall from Theorem 5.3.6 that $\text{ind}_{\widetilde{B}}^{\widetilde{G}}\rho = V_1 \oplus V_2$, where the V_i are two inequivalent subrepresentations of $\text{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$. Let $\hat{V}_1$ and $\hat{V}_2$ be the corresponding irreducible subspaces in the λ-realization of $\text{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$. The next proposition shows that each $\hat{V}_i$, $i \in \{1, 2\}$ corresponds to either E_1 or E_2.

Proposition 6.1.21. *Let $i \in \{1, 2\}$; there exists $j \in \{1, 2\}$ such that for all $\mathfrak{h} \in \hat{V}_i$, $\mathfrak{h}(1) \in E_j$.*

Proof: Let $i \in \{1,2\}$; since $\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho) = \hat{V}_1 \oplus \hat{V}_2$, there exists an intertwining operator $\mathsf{A} \in \mathrm{End}_{\widetilde{G}}(\mathcal{F} \circ \boldsymbol{\eta}(\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho))$ such that $\ker(\mathsf{A}) = \hat{V}_i$. We write $\mathsf{A} = \mathrm{T}_{\mathsf{a}}$, as in Lemma 6.1.9. Then by Theorem 6.1.19, $\mathsf{a} = \alpha \mathrm{I}_n + \beta B$, for some $\alpha, \beta \in \mathbb{C}$. Let $\mathfrak{h} \in \hat{V}_i$. Then $\mathsf{A}(\mathfrak{h})(u) = \mathsf{a}(u)\mathfrak{h}(u) = 0$, $\forall u \in \mathbb{F}$. In particular, $\mathsf{a}(1)\mathfrak{h}(1) = 0$, whence by Theorem 6.1.19, with $\beta \neq 0$, $h(1) = -\alpha/\beta h(1)$. Therefore, $h(1)$ is an eigenvector for B and by Theorem 6.1.19, there exists $j \in \{1,2\}$, such that $\mathfrak{h}(1)$ is in E_j. ∎

Remark 6.1.22. We choose to denote the irreducible component corresponding to the eigenspace E_1 by $\hat{V}_1$ and the one to the eigenspace E_2 by $\hat{V}_2$.

Because the eigenvalues of B are 0 and $q^{-\frac{1}{2}} + q^{\frac{1}{2}}$, it follows that the two possible forms of $\mathsf{A} = \mathrm{T}_{\mathsf{a}}$ satisfy $\mathsf{a}(1) = \beta B$ or $\mathsf{a}(1) = \beta B - (q^{-\frac{1}{2}} + q^{\frac{1}{2}})\mathrm{I}_n$ for some $\beta \in \mathbb{C}$. These matrices are illustrated in the following examples.

Example 6.1.23. *For $n = 3$,*

$$a(1) = \beta \begin{pmatrix} q^{-\frac{1}{2}} & 0 & C_1^{-1} \\ 0 & 0 & 0 \\ C_1 & 0 & q^{\frac{1}{2}} \end{pmatrix}, \quad \text{or} \quad a(1) = \beta \begin{pmatrix} -q^{\frac{1}{2}} & 0 & C_1^{-1} \\ 0 & -(q^{-\frac{1}{2}} + q^{\frac{1}{2}}) & 0 \\ C_1 & 0 & -q^{-\frac{1}{2}} \end{pmatrix},$$

where $\beta \in \mathbb{C}$. Let $i \in \{1,2\}$; then for every $\mathfrak{h} \in \hat{V}_i$, we have

$$\mathfrak{h}(1) \in \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{q}C_2 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \text{or} \quad \mathfrak{h}(1) \in \mathrm{Span}\left\{ \begin{pmatrix} \frac{C_2}{\sqrt{q}} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Example 6.1.24. *For $n = 5$, we obtain*

$$\mathsf{a}(1) = \beta \begin{pmatrix} q^{-\frac{1}{2}} & 0 & 0 & 0 & C_4 \\ 0 & q^{-\frac{1}{2}} & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & C_2^{-1} & 0 & q^{\frac{1}{2}} & 0 \\ C_4^{-1} & 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix},$$

or

$$\mathsf{a}(1) = \beta \begin{pmatrix} -q^{\frac{1}{2}} & 0 & 0 & 0 & C_4 \\ 0 & -q^{\frac{1}{2}} & 0 & C_2 & 0 \\ 0 & 0 & -(q^{-\frac{1}{2}} + q^{\frac{1}{2}}) & 0 & 0 \\ 0 & C_2^{-1} & 0 & -q^{-\frac{1}{2}} & 0 \\ C_4^{-1} & 0 & 0 & 0 & -q^{-\frac{1}{2}} \end{pmatrix},$$

and

$$E_1 = \mathrm{Span}\left\{ \begin{pmatrix} -q^{\frac{1}{2}}C_4 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -q^{\frac{1}{2}}C_2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$E_2 = \mathrm{Span}\left\{ \begin{pmatrix} C_4 q^{-\frac{1}{2}} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ C_2 q^{-\frac{1}{2}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

6.2 Distribution of K-Types

In this section, our goal is to investigate the distribution of K-types of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$. Note that the unramified character $\chi = \chi_{\mathbf{s}}$, $\mathbf{s} = \frac{\pi i}{\log q}$ is primitive mod one. Recall from Section 5.1.1 that χ_0 is a fixed extension of χ to A, and that $\chi_0|_{\widetilde{T}\cap\widetilde{K}}(\mathrm{dg}(t), \zeta) = \epsilon(\zeta)$, for all $t \in \mathcal{O}^\times$ and $\zeta \in \mu_n$. Moreover, recall from Lemma 4.2.1 that the χ_i, $0 \leq i < n$, are characters of $\widetilde{T} \cap \widetilde{K}$ defined by

$$\chi_i(\mathrm{dg}(t), \zeta) = \epsilon\left(\zeta \vartheta(t)^{2i}\right),$$

for $t \in \mathcal{O}^\times$ and $\zeta \in \mu_n$. It follows from Theorem 4.4.19 that

$$\mathrm{Res}_{\widetilde{K}}\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}\rho \cong \bigoplus_{i=0}^{n-1}\left((\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_k)^{K_1}\right) \oplus \bigoplus_{l>1}\left(\widetilde{W}_{0,l}^{+} \oplus \widetilde{W}_{0,l}^{-}\right)^{\oplus n}, \tag{6.2.1}$$

using the notation in Section 4.2. Because $\chi_0|_{(T\cap K)^2}$ is trivial, by Theorem 4.4.19, $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{K_1}$ appears in (6.2.1) with multiplicity one, and decomposes into two irreducible constituents.

Evidently, one of the irreducible constituents of $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0\right)^{K_1}$ is $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0\right)^{\widetilde{K}_0}$, a $\widetilde{K}_0$-invariant one-dimensional subspace, which we call the spherical K-type. We call the complement of $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{\widetilde{K}_0}$ in $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{K_1}$ the *Steinberg* representation, because of the close relationship to the Steinberg representation of finite groups, and denote it by St. Next, we pick a function in each of these $\widetilde{K}$-spaces, and calculate their images under $\mathcal{F} \circ \boldsymbol{\eta}$, evaluated at 1; then we determine to which of the eigenspaces E_1 or E_2, given in Lemma 6.1.20, they belong. This allows us to determine the irreducible subrepresentation of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ to which the $\widetilde{K}$-spaces belong.

6.2.1 The Spherical K-Type

In this section, we will determine the subrepresentation of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ to which $(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{\widetilde{K}_0}$ belongs. Recall that f_i, $0 \le i < n$, is a function in $\mathrm{Ind}_A^{\widetilde{T}}\chi$ whose support is in the coset $A\iota(\varpi^i)$, and $f_i(\iota(\varpi^i)) = 1$. Observe that we can identify f_i with $e_{i+1} \in \mathbb{C}^n$, $0 \le i < n$, via the map in (6.1.1). Recall from Section 5.1.2 that

$$\bar{\pi}^{\widetilde{K}_0} = (\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0)^{\widetilde{K}_0} = \mathrm{Span}\{\phi\},$$

where the normalized spherical function ϕ is given by

$$\begin{aligned} \phi : \widetilde{G} &\rightarrow \mathrm{Ind}_A^{\widetilde{T}}\chi \\ \mathtt{ntk} &\mapsto |\mathtt{t}|\rho(\mathtt{t})f_0, \end{aligned}$$

for $\mathtt{n} \in N$, $\mathtt{t} \in \widetilde{T}$ and $\mathtt{k} \in \widetilde{K}_0$.

Note that ϕ, and hence $\bar{\pi}^{\widetilde{K}_0}$, lies in one of the two irreducible components of $\operatorname{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$. By Proposition 6.1.21, we can determine which of the $\widetilde{G}$-subspaces V_1 and V_2 the function ϕ belongs to, by determining which of the eigenspaces E_1 and E_2 the $(\mathcal{F} \circ \eta_\phi)(1)$ belongs to. Over the course of the next few lemmas, we will calculate $(\mathcal{F} \circ \eta_\phi)(1)$.

Lemma 6.2.1. *Let $\boldsymbol{\eta}$ be the map defined in* (6.1.5) *and ϕ be the normalized spherical function. Then,*

$$\eta_\phi(x) = \begin{cases} f_0, & \textit{if } \operatorname{val}(x) \geq 0 \\ (-1)^{\lfloor \frac{\operatorname{rsd}_{2n}(\operatorname{val}(x))}{n} \rfloor} q^{\operatorname{val}(x)} \epsilon\left(\vartheta(x)^{\operatorname{val}(x)}\right) f_{\operatorname{rsd}_n(\operatorname{val}(x))}, & \textit{if } \operatorname{val}(x) < 0. \end{cases}$$

Proof: Recall from the definition of $\boldsymbol{\eta}$ (6.1.5) in Section 6.1.1 that $\eta_\phi(x) = \phi\left(\operatorname{lt}(x), 1\right)$. Let $x = x_0 \varpi^m$ for $x_0 \in \mathcal{O}^\times$ and $m \in \mathbb{Z}$. We have two possibilities:

- If $x \in \mathcal{O}$, then $(\operatorname{lt}(x), 1) \in \widetilde{K}_0$, and because ϕ is fixed by $\widetilde{K}_0$

$$\phi\left(\operatorname{lt}(x), 1\right) = \phi(1) = f_0.$$

- If $x \notin \mathcal{O}$, then $x^{-1} \in \mathcal{O}$ and hence, $(\operatorname{ut}(x^{-1}), 1) \in \widetilde{K}_0$. Recall from the decomposition in (5.1.5) that for all $x \in \mathbb{F}$,

$$(\operatorname{lt}(x), 1) = \left(\operatorname{ut}(x^{-1}), 1\right) \iota(-x^{-1}) \widetilde{w} \left(\operatorname{ut}(x^{-1}), 1\right). \tag{6.2.2}$$

  Therefore, $\phi\left(\operatorname{lt}(x), 1\right) = |-x^{-1}| \rho\left(\iota(-x^{-1})\right) f_0$. Observe that $|-x^{-1}| = q^m$ with $m < 0$, and

$$\begin{aligned} \rho\left(\iota(-x^{-1})\right) f_0 &= \rho\left(\left(\operatorname{dg}(\varpi^{-m}), \vartheta(x_0)^m\right) \iota(-x_0^{-1})\right) f_0 \\ &= \epsilon\left(\vartheta(x_0)^m\right) \rho\left(\iota(\varpi^{-m})\right) \rho\left(\iota(-x_0^{-1})\right) f_0. \end{aligned}$$

  Recall from Lemma 5.1.5 that $\rho\left(\iota(-x_0^{-1})\right) f_0 = f_0$. Note that $\rho\left(\iota(\varpi^{-m})\right) = \rho\left(\iota(\varpi)\right)^{-m}$ is the matrix calculated in (6.1.4). Observe that $\rho\left(\iota(\varpi^{-m})\right) f_0$ is the first column of $\rho\left(\iota(\varpi^{-m})\right)$. It follows directly from (6.1.4) that the first column of $\rho\left(\iota(\varpi^{-m})\right)$ is $(-1)^{\lfloor \frac{\operatorname{rsd}_{2n}(m)}{n} \rfloor} f_{\operatorname{rsd}_n(m)}$.

Because n is odd, $\vartheta(x_0) = \vartheta(x)$, where $x = x_0\varpi^m$ for $x_0 \in \mathcal{O}^\times$. Hence,

$$\eta_\phi = \phi(\mathrm{lt}(x), 1) = (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(m)}{n} \rfloor} q^m \epsilon\left(\vartheta(x)^m\right) f_{\mathrm{rsd}_n(m)}.$$

∎

As in Section 6.1.3, set $L = \{x \in \mathbb{F}^\times \mid n|\mathrm{val}(x)\}$.

Lemma 6.2.2. *Let $\widehat{\varphi} = \mathcal{F} \circ \eta_\phi$ be the λ-realization of ϕ. Then*

$$\widehat{\varphi}(1) = f_0 - C_1 q^{-\frac{1}{2}} f_{n-1}.$$

Proof: Recall that $\lambda|_{\mathcal{O}} = 1$. Therefore, using Lemma 6.2.1,

$$\widehat{\varphi}(1) = \int_{\mathbb{F}} \lambda(-x)\eta_\phi(x)dx = \int_{\mathcal{O}} f_0 dx$$
$$+ \sum_{i=0}^{n-1} \int_{\{x \in \varpi^i L | \mathrm{val}(x) < 0\}} (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} \lambda(-x) q^{\mathrm{val}(x)} \epsilon\left(\vartheta(x)^{\mathrm{val}(x)}\right) f_i dx.$$

Observe that

$$\int_{\{x \in \mathbb{F} | \mathrm{val}(x) = m\}} \lambda(-x)(-q)^{\mathrm{val}(x)} \epsilon\left(\vartheta(x)^\alpha\right) dx = (-q)^m \int_{\mathcal{O}^\times} \lambda(-x_0\varpi^m) \epsilon\left(\vartheta(x)^\alpha\right) dx. \tag{6.2.3}$$

It follows from Lemma 1.1.18 that for $\alpha \in \{1, \cdots, n-1\}$, the integral (6.2.3) is only non-zero when $m = -1$. Therefore, the above sum can be simplified to

$$\begin{aligned} \widehat{\varphi}(1) = f_0 \quad &+ \int_{\{x \in L | \mathrm{val}(x) < 0\}} (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} \lambda(-x)(q)^{\mathrm{val}(x)} f_0 dx \\ &- \int_{\{x \in \mathbb{F} | \mathrm{val}(x) = -1\}} \lambda(-x)(\frac{1}{q}) \epsilon\left(\vartheta(x)^{n-1}\right) f_{n-1} dx. \end{aligned}$$

Let us calculate the second and third summand of the above expression separately. For the second term, observe that

$$\int_{\{x \in L | \mathrm{val}(x) < 0\}} (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} \lambda(-x)(q)^{\mathrm{val}(x)} dx = \lim_{r \to -\infty} \sum_{k=-1}^{r} (-q)^{nk} \int_{\{x \in L \mid \mathrm{val}(x) = nk\}} \lambda(-x) dx.$$

For each $k < 0$, by Lemma 1.1.12 we have

$$\int_{\{x\in L \mid \mathrm{val}(x)=nk\}} \lambda(-x)dx = \int_{\mathfrak{p}^{nk}} \lambda(-x)dx - \int_{\mathfrak{p}^{nk+1}} \lambda(-x)dx = 0.$$

Now, let us calculate the third term.

$$\begin{aligned}
& \int_{\{x\in\mathbb{F}|\mathrm{val}(x)=-1\}} \lambda(-x)(\frac{1}{q})\epsilon\left(\vartheta(x)^{n-1}\right)dx \\
= & \int_{\{x\in\mathbb{F}|\mathrm{val}(x)=-1\}} \lambda(-x)\epsilon\left(\vartheta(x)^{-1}\right)|x|^{-1}dx \\
= & \int_{\{x\in\mathbb{F}|\mathrm{val}(x)=-1\}} \lambda(x)\epsilon\left(\vartheta(x)^{-1}\right)|x|^{-1}dx \quad \text{change of variable} \\
= & \; q^{-\frac{1}{2}}\int_{\{x\in\mathbb{F}|\mathrm{val}(x)=-1\}} \lambda(x)\epsilon\left(\vartheta(x)^{-1}\right)|x|^{\frac{1}{2}}|x|^{-1}dx \\
= & \; q^{-\frac{1}{2}}\Gamma_{(x,\varpi)_n}(\frac{1}{2}) \quad \text{by Definition 1.1.17} \\
= & \; q^{-\frac{1}{2}}C_1 \quad \text{by Theorem 1.1.19.}
\end{aligned}$$

Therefore,

$$\widehat{\varphi}(1) = f_0 - C_1 q^{-\frac{1}{2}} f_{n-1}.$$

∎

Theorem 6.2.3. *The $\widetilde{K}$-irreducible representation $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0\right)^{\widetilde{K}_0}$ is a $\widetilde{K}$-subrepresentation of V_1.*

Proof: The space $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0\right)^{\widetilde{K}_0}$ is generated by the spherical vector ϕ. By Lemma 6.2.2, the λ-realization of ϕ evaluated at 1 is a scalar multiple of $-e_1C_1^{-1}q^{\frac{1}{2}} + e_n$, that is, by Lemma 6.1.20, a vector in the basis of E_1. Hence, by Lemma 6.1.21, $\left(\mathrm{Ind}_{\widetilde{B}\cap\widetilde{K}}^{\widetilde{K}}\chi_0\right)^{\widetilde{K}_0}$ is a subspace of V_1. ∎

6.2.2 The Steinberg K-Type

Recall that

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \mid a, b, d \in \mathcal{O},\ c \in \mathfrak{p} \right\}$$

is the Iwahori subgroup of G, and $\widetilde{I}$ is its inverse image in $\widetilde{G}$. Note that since for all $\mathtt{g} \in \widetilde{G}$, we can write $\mathtt{g} = \mathtt{ntk}$ for some $\mathtt{n} \in N, \mathtt{t} \in \widetilde{T}$ and $\mathtt{k} \in \widetilde{K}_0$, and $f(\mathtt{g}) = |\mathtt{t}|\rho(\mathtt{t})f(\mathtt{k})$, for every $f \in \operatorname{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$, the value of f over $\widetilde{K}_0$ determines the function completely. Define the function $\psi \in \operatorname{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ by

$$\psi(\mathtt{k}) = \begin{cases} f_0, & \mathtt{k} \in \widetilde{I} \cap \widetilde{K}_0 \\ -f_0, & \mathtt{k} \in \left(w\widetilde{I}\right) \cap \widetilde{K}_0 \\ 0, & \text{otherwise,} \end{cases} \tag{6.2.4}$$

where $k \in \widetilde{K}_0$. It is not difficult to see that ψ is in St. Next, by calculating $\mathcal{F} \circ \eta_\psi$, we will determine to which irreducible component of $\operatorname{ind}_{\widetilde{B}}^{\widetilde{G}}\rho$ the function ψ belongs.

Lemma 6.2.4. *Let $\boldsymbol{\eta}$ be the map defined in* (6.1.5) *and ψ be as in* (6.2.4)*. Then,*

$$\eta_\psi(x) = \begin{cases} f_0, & \mathrm{val}(x) > 0 \\ 0, & \mathrm{val}(x) = 0 \\ -(-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} q^{\mathrm{val}(x)} \epsilon\left(\vartheta(x)^{\mathrm{val}(x)}\right) f_{\mathrm{rsd}_n(\mathrm{val}(x))}, & \mathrm{val}(x) < 0. \end{cases} \tag{6.2.5}$$

Proof: Let $x = x_0\varpi^m$ for $x_0 \in \mathcal{O}^\times$ and $m \in \mathbb{Z}$. We have two possibilities:

- If $x \in \mathcal{O}$, then by (6.2.4)

$$\psi\left(\mathrm{lt}(x), 1\right) = \begin{cases} f_0, & \text{if } \mathrm{val}(x) > 0 \\ 0, & \text{if } \mathrm{val}(x) = 0 \end{cases}$$

- If $x \notin \mathcal{O}$ then $x^{-1} \in \mathcal{O}$ and hence, $(\mathrm{ut}(x^{-1}), 1) \in K_1$. Therefore, by decomposition (5.1.5)

$$\psi\left(\mathrm{lt}(x), 1\right) \;\; = \;\; \psi\left(\left(\mathrm{ut}(x^{-1}), 1\right)\iota(-x^{-1})\widetilde{w}\right)$$

$$= |-x^{-1}|\rho(\iota(-x^{-1}))\psi(\widetilde{w})$$
$$= -|x^{-1}|\rho(\iota(-x^{-1}))f_0.$$

Observe that $|x^{-1}| = q^m$, with $m < 0$ and

$$\begin{aligned} \rho(\iota(-x^{-1}))f_0 &= \epsilon\left(\vartheta(x_0)^m\right)\rho\left(\iota(\varpi^{-m})\right)\rho\left(\iota(-x_0^{-1})\right)f_0 \\ &= \epsilon\left(\vartheta(x_0)^m\right)\rho\left(\iota(\varpi^{-m})\right)f_0. \end{aligned}$$

Hence, because by (6.1.4) we have $\rho\left(\iota(\varpi^{-m})\right)f_0 = (-1)^{\lfloor\frac{\mathrm{rsd}_{2n}(m)}{n}\rfloor}f_{\mathrm{rsd}_n(m)}$, we obtain

$$\eta_\psi(x) = -(-1)^{\lfloor\frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n}\rfloor}q^{\mathrm{val}(x)}\epsilon\left(\vartheta(x)^{\mathrm{val}(x)}\right)f_{\mathrm{rsd}_n(\mathrm{val}(x))}.$$

∎

Lemma 6.2.5. *Let $\widehat{\psi}$ be the λ realization of ψ. Then*

$$\widehat{\psi}(1) = q^{-1}f_0 + q^{-\frac{1}{2}}C_1f_{n-1}.$$

Proof: It follows from Lemma 6.2.4 that

$$\begin{aligned} \widehat{\psi}(1) &= \int_{\mathbb{F}}\lambda(-x)\eta_\psi(x)dx \\ &= \int_{\mathcal{O}}\lambda(-x)\eta_\psi(x)dx \\ &\quad -\sum_{i=0}^{n-1}\int_{\{x\in\varpi^iL|\mathrm{val}(x)<0\}}(-1)^{\lfloor\frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n}\rfloor}\lambda(-x)q^{\mathrm{val}(x)}\epsilon\left(\vartheta(x)_n^{\mathrm{val}(x)}\right)f_idx. \end{aligned}$$

Observe that

$$\begin{aligned} \int_{\mathcal{O}}\lambda(-x)\eta_\psi(x)dx &= \int_{\mathcal{O}^\times}0dx + \int_{\mathfrak{p}}f_0dx \\ &= \mu(\mathfrak{p})f_0 \\ &= q^{-1}f_0, \end{aligned}$$

and it follows from Lemma 1.1.18 that for $\alpha \in \{1, \cdots, n-1\}$,

$$\int_{\{x\in\mathbb{F}|\mathrm{val}(x)=m\}} (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} \lambda(-x) q^{\mathrm{val}(x)} \epsilon\left((\vartheta(x)^{\alpha}\right) f_i dx$$

is only nonzero when $m = -1$. Therefore, $\widehat{\psi}(1)$ can be simplified to

$$\begin{aligned} \widehat{\varphi}(1) = q^{-1} f_0 \quad &- \quad \int_{\{x\in L|\mathrm{val}(x)<0\}} (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} \lambda(-x)(q)^{\mathrm{val}(x)} f_0 dx \\ &+ \quad \int_{\{x\in\mathbb{F}|\mathrm{val}(x)=-1\}} \lambda(-x)(\frac{1}{q})\epsilon\left(\vartheta(x)^{n-1}\right) f_{n-1} dx. \end{aligned}$$

Lemma 1.1.12 implies that

$$\int_{\{x\in L|\mathrm{val}(x)<0\}} (-1)^{\lfloor \frac{\mathrm{rsd}_{2n}(\mathrm{val}(x))}{n} \rfloor} \lambda(-x)(q)^{\mathrm{val}(x)} f_0 \, dx = 0.$$

Plus,

$$\begin{aligned} &\int_{\{x\in\mathbb{F} \mid \mathrm{val}(x)=-1\}} \lambda(-x) q^{-1} \epsilon\left(\vartheta(x)^{n-1}\right) f_{n-1} dx \\ = q^{-\frac{1}{2}} &\int_{\{x\in\mathbb{F} \mid \mathrm{val}(x)=-1\}} \lambda(-x) q^{\frac{1}{2}} \epsilon\left(\vartheta(x)^{n-1}\right) f_{n-1} \frac{dx}{|x|} = q^{-\frac{1}{2}} \Gamma_{(x,\varpi)_n}(\frac{1}{2}) f_{n-1} \\ &= q^{-\frac{1}{2}} C_1 f_{n-1}. \end{aligned}$$

Hence,

$$\int_{\mathbb{F}} \lambda(-x)\eta_{\psi}(x)dx = q^{-1} f_0 + q^{-\frac{1}{2}} C_1 f_{n-1}.$$

∎

Theorem 6.2.6. *The $\widetilde{K}$-irreducible representation* St *is a $\widetilde{K}$-subrepresentation of V_2.*

Proof: By Lemma 6.2.5, the λ-realization of $\psi \in$ St evaluated at 1 is a scalar multiple of $e_1 C_1^{-1} q^{-\frac{1}{2}} + e_n$, and therefore, by Lemma 6.1.20, a vector in the basis of E_2. Hence, by Lemma 6.1.21, St is a subspace of V_2. ∎

We have determined the distribution of the two irreducible constituents of the first level representations in the K-type decomposition in (6.2.1). With a similar technique, we can determine the irreducible constituent to which the rest of the first level representations in the K-type belong. Further, we hope to determine the K-type distribution completely in the interests of describing the irreducible constituent of $\mathrm{ind}_{\widetilde{B}}^{\widetilde{G}}$.

Part II

Faithful Representations of Chevalley Groups over $\mathcal{O}/\mathfrak{p}$

Chapter 7

Minimal Degree Problem

In this chapter, in addition to basic knowledge of local fields, we assume knowledge of Lie algebra and Chevalley groups, as can be found in [Hum78] and [Ste68].

Given a finite group G, the minimal degree problem is to find (a lower bound for) the smallest possible dimension of a representation of G. This chapter presents a joint paper with Mohammad Bardestani, Keivan Mallahi-Karai and Hadi Salmasian, in which we consider the minimal degree problem for a family of Chevalley groups, as described in the abstract:

> Let F be a non-Archimedean local field with the ring of integers $\mathcal{O}$ and the prime ideal $\mathfrak{p}$ and let $G = \mathbf{G}_{ad}\left(\mathcal{O}/\mathfrak{p}^n\right)$ be the adjoint Chevalley group. Let $m_{\mathfrak{f}}(G)$ denote the smallest possible dimension of a faithful representation of G. Using the Stone-von Neumann theorem, we determine a lower bound for $m_{\mathfrak{f}}(G)$ which is asymptotically the same as the results of Landazuri, Seitz and Zalesskii for split Chevalley groups over $\mathbb{F}_q$. Our result yields a conceptual explanation of the exponents that appear in the aforementioned results.([BKMKS15, Abstract])

The history and applications of the problem can be found in Section 7.1. Instead, here I mention the link between the ideas in this paper and Part I of the thesis. First of all, the groups considered in this chapter are finite and hence, their representation theory follows that reviewed in Chapter 1. The main idea of this paper is to use the Stone-von Neumann theorem to calculate the dimension of certain subrepresentations of a given faithful representation. Namely, given a faithful representation (ρ, V) of (an adjoint) Chevalley group G over $\mathcal{O}/\mathfrak{p}^n$, we construct a Heisenberg subgroup U of G and show that the decomposition of $\rho|_U$ into irreducible representations contains at least one constituent (σ_1, V_1), to which the Stone-von Neumann theorem applies. Hence, the unique (up to isomorphism) construction of V_1 allows us to compute the dimension of V_1. The lower bound is then obtained by calculating the size of the orbit of the action of the normalizer of U on (σ_1, V_1).

As it is with the nature of a joint work, most of the sections were written jointly. My role in this paper was mainly centred on the construction of the Heisenberg subgroup of Chevalley groups, and applying the Stone-von Neumann theorem, that is Sections 7.3, 7.4, 7.5, and 7.6, which led to calculating the dimension of the subrepresentation V_1.

The rest of the chapter is a verbatim copy of the paper titled "Faithful representations of Chevalley groups over quotient rings of non-Archimedean local fields".

7.1 Introduction

For a finite group G, let $\mathrm{Rep}_{\mathfrak{f}}(G)$ denote the set of all finite-dimensional faithful representations of G over complex vector spaces, and set

$$m_{\mathfrak{f}}(G) := \min\{d_\rho \,:\, \rho \in \mathrm{Rep}_{\mathfrak{f}}(G)\},$$

where d_ρ denotes the dimension (also called the degree) of ρ. Lower bounds on $m_{\mathfrak{f}}(G)$ can be found in group theory literature as old as the work of Frobenius [Fro]. Indeed, by constructing the character table of $\mathrm{PSL}_2(\mathbb{F}_p)$, Frobenius showed that

$$m_{\mathfrak{f}}\left(\mathrm{PSL}_2(\mathbb{F}_p)\right) \geq \frac{p-1}{2} \text{ for every prime } p \geq 5.$$

Apart from its intrinsic interest, the Frobenius bound has applications in many questions in number theory and additive combinatorics. To name a few, Sarnak and Xue [SX91] were the first to use this bound to obtain a lower bound for the smallest non-trivial eigenvalue of the Laplace-Beltrami operator on the hyperbolic space. This idea was subsequently used by Bourgain and Gamburd [BG08b] to answer the 1-2-3 question of Lubotzky on the uniform expansion bounds for the Cayley graphs of $\mathrm{SL}_2\left(\mathbb{F}_p\right)$.

The Frobenius bound has been generalized by Landazuri, Seitz and Zalesskii [LS74, SZ93] to other families of finite simple groups of Lie type. These bounds play an essential role in the theory of expander graphs and approximate groups [BGT11, Lub12]. The finite simple groups of Lie type are canonically obtained by reduction mod $\mathfrak{p}$ of the group $\mathbf{G}(\mathcal{O})$ of $\mathcal{O}$-points of a Chevalley group, where $\mathcal{O}$ is the ring of integers of a local field. However, despite interesting applications (see below), little work has been

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done to extend the aforementioned bounds to reduction mod $\mathfrak{p}^n$ of $\mathbf{G}(\mathcal{O})$. Bourgain and Gamburd [BG08a] considered this problem for $\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})$ in order to show that, for any sufficiently large prime p and any symmetric set S generating a Zariski-dense subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the family of Cayley graphs $\{\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z}), \pi_{p^n}(S))\}_{n\geq 1}$ is an expander family (here π_{p^n} is the reduction map modulo p^n). Indeed, they proved

$$m_{\mathsf{f}}(\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})) \geq \frac{p^{n-2}(p^2-1)}{2} \quad \text{for } n \geq 2.$$

Motivated by the works of Bourgain and Gamburd mentioned above, the first and third authors of this paper studied $m_{\mathsf{f}}(\mathrm{SL}_k(\mathbb{Z}/p^n\mathbb{Z}))$ and $m_{\mathsf{f}}(\mathrm{Sp}_{2k}(\mathbb{Z}/p^n\mathbb{Z}))$ [BMK15]. The same problem for $m_{\mathsf{f}}(\mathrm{SL}_k(\mathbb{Z}/p^n\mathbb{Z}))$ has been considered by de Saxcé [dS13].

Let F be a non-Archimedean local field with the ring of integers $\mathcal{O}$ and the prime ideal $\mathfrak{p}$. The order of residue field $\mathcal{O}/\mathfrak{p}$ is denoted by $q = p^l$ where p is a prime number. Our aim in this paper is to obtain a bound for the minimal dimension of all faithful complex representations of adjoint Chevalley groups over the ring $\mathcal{O}/\mathfrak{p}^n$ where n is a positive integer. Indeed for the Chevalley group $\mathbf{G}_{ad}(\mathcal{O}/\mathfrak{p}^n)$ associated to a simple Lie algebra $\mathfrak{g}$ we will obtain the following bound

$$m_{\mathsf{f}}(\mathbf{G}_{ad}(\mathcal{O}/\mathfrak{p}^n)) \geq Cq^{\frac{n(r+1)}{2}}, \tag{7.1.1}$$

where r is the dimension of the nilpotent radical of the Heisenberg parabolic subalgebra of $\mathfrak{g}$, and $C > 0$ is an absolute constant (independent of q and r).

We remark that all adjoint Chevalley groups over $\mathcal{O}/\mathfrak{p}$ (except for a few cases) are simple groups, and so all of their non-trivial representations are faithful. Therefore, our Theorem 7.1.1 below for $n = 1$ yields lower bounds for non-trivial representations of $\mathbf{G}_{ad}(\mathbb{F}_q)$, which are asymptotically the same as the results of [LS74, SZ93] for split Chevalley groups over $\mathbb{F}_q$. By being asymptotically the same we mean that the exponents that appear in (7.1.1) are the same as those in the work of Landazuri, Seitz and Zalesskii. For a corrigendum to [LS74] the cases $\mathsf{F}_4(q)$, q odd, and ${}^2\mathsf{E}_6(q)$, we refer the reader to [SZ93].

Let us briefly sketch the idea of this paper. For a given simple Lie algebra $\mathfrak{g}$, following Gross and Wallach's idea [GW96], we consider its Heisenberg parabolic subalgebra whose nilpotent radical is a two step nilpotent subalgebra. This nilpotent subalgebra gives rise to a two step nilpotent subgroup of the adjoint Chevalley group associated to $\mathfrak{g}$ (this is the general $2n+1$-dimensional Heisenberg group for some n). The irreducible representations of a Heisenberg group are classified by their central characters via the Stone-von Neumann theorem. Given a faithful representation ρ of $\mathbf{G}_{ad}(\mathcal{O}/\mathfrak{p}^n)$, we consider its restriction to the aforementioned Heisenberg subgroup and we find the polarizing subgroup of the generic character of the center. Our bound is then obtained by orbit counting of the action of a certain subgroup on an irreducible component of the representation ρ. Our main theorem is the following:

Theorem 7.1.1. *Let F be a non-Archimedean local field with the ring of integers $\mathcal{O}$, prime ideal $\mathfrak{p}$, and residue field $\mathcal{O}/\mathfrak{p} \cong \mathbb{F}_q$, where $q = p^l$ for a prime number p. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with root system Φ. Let*

$$\mathbf{G}_{ad}\left(\mathcal{O}/\mathfrak{p}^n\right) := \mathbf{G}_{ad}\left(\mathcal{O}/\mathfrak{p}^n, \Phi\right)$$

be the adjoint Chevalley group associated to $\mathfrak{g}$. Then $m_{\mathfrak{f}}\left(\mathbf{G}_{ad}\left(\mathcal{O}/\mathfrak{p}^n\right)\right) \geq h_{\mathfrak{f}}(\Phi, q, n)$, *where* $h_{\mathrm{f}}(\Phi, q, n)$ *is given in the following table:*

Φ		$h_{\mathrm{f}}(\Phi, q, n)$
A_1	$p \geq 3$	$\frac{1}{2}\left(q^n - q^{n-1}\right)$
A_m	$m \geq 2$, $p \geq 3$	$\left(q^n - q^{n-1}\right) q^{(m-1)n}$
B_m	$m \geq 3$, $p \geq 3$	$\left(q^n - q^{n-1}\right) q^{(2m-3)n}$
C_m	$m \geq 2$, $p \geq 3$	$\frac{1}{2}\left(q^n - q^{n-1}\right) q^{(m-1)n}$
D_m	$m \geq 4$, $p \geq 3$	$\left(q^n - q^{n-1}\right) q^{(2m-4)n}$
G_2	$p \geq 5$	$\left(q^n - q^{n-1}\right) q^{2n}$
F_4	$p \geq 3$	$\left(q^n - q^{n-1}\right) q^{7n}$
E_6	$p \geq 3$	$\left(q^n - q^{n-1}\right) q^{10n}$
E_7	$p \geq 3$	$\left(q^n - q^{n-1}\right) q^{16n}$
E_8	$p \geq 3$	$\left(q^n - q^{n-1}\right) q^{28n}$

Remark 7.1.2. Ree [Ree57] (see also [Car89], Theorem 11.3.2) proved that the groups $\mathbf{G}_{ad}(\mathbb{F}_q)$ are indeed what one would expect to obtain, namely, $\mathrm{PSL}_m(\mathbb{F}_q)$ if $\mathfrak{g}$ is of type A_{m-1}; $\mathrm{PSp}_{2m}(\mathbb{F}_q)$ if $\mathfrak{g}$ is of type C_m; $\mathrm{P}\Omega_{2\mathrm{m}}(\mathbb{F}_q)$ if $\mathfrak{g}$ is of type D_m and $\mathrm{P}\Omega_{2m+1}(\mathbb{F}_q)$ if $\mathfrak{g}$ is of type B_m and $q \geq 3$.

Remark 7.1.3. For simplicity of presentation we just consider the *adjoint* Chevalley groups. However, the result can also be extended to the simply connected Chevalley groups. Chevalley proved that the group $\mathbf{G}_{ad}(\mathbb{F}_q)$ is simple except for $\mathsf{A}_1(2)$, $\mathsf{A}_1(3)$, $\mathsf{B}_2(2)$ and $\mathsf{G}_2(2)$ (see [Car89], Theorem 11.1.2).

Remark 7.1.4. Let us point out that the idea of restriction to nilpotent subgroups was also used in [LS74]. Nevertheless, the arguments in [LS74] are long and case by case. One of our main goals in writing this paper is to give a uniform argument based on the idea of Heisenberg parabolic subalgebras to obtain lower bounds which, in the special case of $\mathcal{O}/\mathfrak{p}$, are asymptotically the same as those given in [LS74]. (See the discussion after (7.1.1) for a precise meaning.) Such bounds are enough for the existing applications. Another important technical detail that has been worked out in our paper is to verify that many facts about Chevalley groups over fields remain valid for Chevalley groups over rings of our interest (see Section 7.4).

7.2 Notations and preliminaries

In this section we set some notation which will be used throughout this paper. We also recall some basic facts about local fields that can be found in [Neu99, Ser79a].

If X is any set, f any function on X, and $Y \subseteq X$ any subset, then $f|_Y$ is the restriction of f to Y. $|X|$ is the cardinality of a finite set X. We will use the shorthand $\mathbf{e}(x) := \exp(2\pi i x)$. For a given group G, its identity element is denoted by $\mathbf{1}$. Moreover $\mathrm{char}(F)$ is the characteristic of a given field.

By the well-known classification of local fields, any non-Archimedean local field is isomorphic to a finite extension of $\mathbb{Q}_p$ (p is a prime number) or is isomorphic to the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field with $q = p^l$ elements. For a non-Archimedean local field F with the discrete valuation ν, we will denote its ring of integers and its unique prime ideal by $\mathcal{O}$ and $\mathfrak{p}$, respectively. We will also fix a uniformizer $\varpi \in \mathfrak{p}$. For any integer $m \in \mathbb{Z}$, we write

$$\mathfrak{p}^m := \{x \in F : \nu(x) \geq m\}.$$

Then $\mathfrak{p}^m/\mathfrak{p}^{m+n} \cong \mathcal{O}/\mathfrak{p}^n$ as additive groups, for every $m, n \in \mathbb{Z}$ with $n > 0$.

Let n be a positive integer. Our goal in this section is to describe all additive characters of the finite local rings $\mathcal{O}/\mathfrak{p}^n$ using the ring structure.

From now on, if $\mathrm{char}(F) = 0$ we set $E = \mathbb{Q}_p$ and if $\mathrm{char}(F) = p > 0$ we set $E = F$. Now we define

$$\mathrm{Tr} := \mathrm{Tr}_{F/E} : F \to E,$$

the trace map of F over E. The *Dedekind's complementary module*, (or *inverse different*) is defined by

$$\mathcal{O}^* := \{x \in F : \nu(\mathrm{Tr}(sx)) \geq 0 \text{ for all } s \in \mathcal{O}\}.$$

One can show that $\mathcal{O}^*$ is a fractional ideal of F and hence for some $\ell \geq 0$ we have $\mathcal{O}^* = \varpi_F^{-\ell}\mathcal{O} = \mathfrak{p}^{-\ell}$. Throughout this paper ℓ designates this exponent. Note that $\ell = 0$ when $\mathrm{char}(F) > 0$.

We now fix an additive character $\psi : F \to \mathbb{C}^*$ as follows. First assume $\mathrm{char}(F) = 0$. For every $x \in \mathbb{Q}_p$, let n_x be the smallest non-negative integer such that $p^{n_x}x \in \mathbb{Z}_p$. Let $r_x \in \mathbb{Z}$ be such that $r_x \equiv p^{n_x}x \pmod{p^{n_x}}$. It is easy to see that the following map (known as a Tate character)

$$\psi : \mathbb{Q}_p \to \mathbb{C}^*, \qquad x \mapsto \mathbf{e}(r_x/p^{n_x}), \tag{7.2.1}$$

is a non-trivial additive character of $\mathbb{Q}_p$ with the kernel $\mathbb{Z}_p$.

Now assume $F = \mathbb{F}_q((T))$, so that $\mathcal{O} = \mathbb{F}_q[[T]]$ and $\varpi = T$. We now set

$$\psi : \mathbb{F}_q((T)) \to \mathbb{C}^*, \qquad \sum_{i \geq N} a_i T^i \to \mathbf{e}\left(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_{-1})/p\right). \tag{7.2.2}$$

Notice that the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$ is surjective. Hence, $\psi|_{\mathcal{O}} = 1$ but $\psi|_{\mathfrak{p}^{-1}} \neq 1$ (sometimes we say that the conductor of ψ is $\mathcal{O} = \mathbb{F}_q[[T]]$).

Lemma 7.2.1. *Let F be a local field with the ring of integers $\mathcal{O}$ and prime ideal $\mathfrak{p}$. All additive characters of the ring $\mathcal{O}/\mathfrak{p}^n$ are given by*

$$\psi_{\bar{b}} : \mathcal{O}/\mathfrak{p}^n \to \mathbb{C}^*, \qquad x + \mathfrak{p}^n \mapsto \psi(\mathrm{Tr}(bx)),$$

where $\bar{b} = b + \mathfrak{p}^{-\ell} \in \mathfrak{p}^{-(n+\ell)}/\mathfrak{p}^{-\ell}$.

Proof: First assume that $\mathrm{char}(F) = 0$, that is, F is a p-adic field. Let $\bar{b}_1 = b_1 + \mathfrak{p}^{-\ell}$ and $\bar{b}_2 = b_2 + \mathfrak{p}^{-\ell}$ be distinct elements and assume that $\psi_{\bar{b}_1} = \psi_{\bar{b}_2}$. Then for all $x \in \mathcal{O}$ we have $\psi(\mathrm{Tr}((b_1 - b_2)x)) = 1$, which implies that $\mathrm{Tr}((b_1 - b_2)x) \in \mathbb{Z}_p$.

Thus $b_1 - b_2 \in \mathfrak{p}^{-\ell}$, which is a contradiction. This construction provides exactly $|\mathfrak{p}^{-(n+\ell)}/\mathfrak{p}^{-\ell}|$ distinct additive characters. Since $|\mathcal{O}/\mathfrak{p}^n| = |\mathfrak{p}^{-(n+\ell)}/\mathfrak{p}^{-\ell}|$, we are done.

Next assume that $\mathrm{char}(F) > 0$. It is clear that the map $\psi_{\bar{b}}$ is well-defined. Now suppose for some $b \in \mathfrak{p}^{-n}$ we have $\psi(bx) = 1$ for all $x \in \mathcal{O}$. Hence the fractional ideal $b\mathcal{O}$ is a subset of $\ker(\psi)$. Therefore $b \in \mathcal{O}$ since the conductor of ψ is $\mathcal{O}$. This construction provides exactly $|\mathfrak{p}^{-n}/\mathcal{O}|$ distinct additive characters. Since $|\mathcal{O}/\mathfrak{p}^n| = |\mathfrak{p}^{-n}/\mathcal{O}|$, we are done. ∎

7.3 The Stone-von Neumann theorem

In this section, we state a version of Stone-von Neumann theorem that suits our purposes in this paper. The Stone-von Neumann theorem holds in a broader setting [How08, McN12, MVW87]. However, we only present it in the finite group case, which is needed in this paper.

Let U be a finite two step nilpotent group. If A is any subgroup of U containing $Z(U)$, we will denote $\bar{A} := A/Z(U)$. Let $\chi : Z(U) \to \mathbb{C}^*$ be a one-dimensional representation of $Z(U)$. We define a pairing

$$U/Z(U) \times U/Z(U) \to \mathbb{C}^* \ , \ \langle xZ(U), yZ(U)\rangle_\chi := \chi\left([x,y]\right).$$

We call χ a *generic character of* $Z(U)$ if the above pairing is non-degenerate, in the sense that for every $x \in U$, if $\langle xZ(U), yZ(U)\rangle_\chi = 1$ for every $y \in U$, then $x \in Z(U)$. Assuming χ is generic character of $Z(U)$, we say that a subgroup $Z(U) \le A \le U$ is *isotropic* if $\bar{A} \subseteq \bar{A}^\perp$, where

$$\bar{A}^\perp := \left\{ xZ(U) \,:\, \langle xZ(U), yZ(U)\rangle_\chi = 1 \text{ for all } y \in A \right\}.$$

We say that A is *polarizing* if $\bar{A} = \bar{A}^\perp$.

For the next theorem we refer the reader to [Bum97], §4.1.

Theorem 7.3.1 (Stone-von Neumann theorem). *Let U be a finite two step nilpotent group, and let χ be a generic character of $Z(U)$. Then there exists a unique isomorphism class of irreducible representations of U with central character χ. Such a representation may be constructed as follows: Let A be any polarizing subgroup of U, and let $\tilde{\chi}$ be any extension of χ to A. Then the representation $\mathrm{Ind}_A^U(\tilde{\chi})$ is of this class.*

7.4 The Heisenberg parabolic subalgebra

This section is devoted to a rapid review of some basic facts in the theory of simple Lie algebras. We closely follow Gross and Wallach's paper [GW96], Sections 1 and 2 (see also [Sal07], §3).

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\Phi \subseteq \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then, we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{7.4.1}$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [H, x] = \alpha(H)x, \forall H \in \mathfrak{h}\}$. Let $E = \mathrm{Span}_{\mathbb{R}}\{\alpha \mid \alpha \in \Phi\}$. Note that E is equipped with a symmetric positive definite inner product $(\, , \,)$ obtained from the Killing form of $\mathfrak{g}$ via the isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$. For $\alpha, \beta \in \Phi$, set $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$. Let $\beta \neq \pm\alpha$ be two independent roots. Assume that $\|\beta\| \geq \|\alpha\|$. Then the values of $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ are given by Table 7.1 (see [Hum78], Table 1, §9.4).

Table 7.1: Root structure

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$(\|\beta\|/\|\alpha\|)^2$
0	0	undetermined
1	1	1
-1	-1	1
1	2	2
-1	-2	2
1	3	3
-1	-3	3

Let Δ be a base of Φ. Let $\Phi^+ \subseteq \Phi$ the set of positive roots with respect to Δ, and let $\tilde{\beta}$ be the highest root. It is known that $\tilde{\beta}$ is a long root and $m_\alpha \geq n_\alpha$, where $\tilde{\beta} = \sum_{\alpha \in \Delta} m_\alpha \alpha$ and $\gamma = \sum_{\alpha \in \Delta} n_\alpha \alpha$ is any $\gamma \in \Phi$. Given the above notation, we define the *Heisenberg parabolic subalgebra* $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. The Levi subalgebra and the nilpotent radical of $\mathfrak{q}$ are:

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Phi \\ \langle \alpha, \tilde{\beta} \rangle = 0}} \mathfrak{g}_\alpha, \qquad \text{and} \qquad \mathfrak{u} = \bigoplus_{\substack{\alpha \in \Phi \\ \langle \alpha, \tilde{\beta} \rangle > 0}} \mathfrak{g}_\alpha.$$

Lemma 7.4.1. *The inequality $\langle \alpha, \tilde{\beta} \rangle > 0$ implies that $\alpha \in \Phi^+$, and either $\alpha = \tilde{\beta}$ or $\langle \alpha, \tilde{\beta} \rangle = 1$. Moreover, if $\langle \alpha, \tilde{\beta} \rangle = 1$, then $\tilde{\beta} - \alpha \in \Phi^+$ and $\langle \tilde{\beta} - \alpha, \tilde{\beta} \rangle = 1$.*

Proof: If $\langle \alpha, \tilde{\beta} \rangle > 0$ then $\tilde{\beta} - \alpha \in \Phi$ (see [Hum78], Lemma of §9.4). This implication, along with the fact that $\tilde{\beta}$ is the highest root, implies $\alpha \in \Phi^+$.

Note that for any $\alpha \in \Phi^+$, $|\langle \alpha, \tilde{\beta} \rangle| \leq |\langle \tilde{\beta}, \alpha \rangle|$. Assume $\alpha \neq \tilde{\beta}$. Then by applying Table 7.1 and a simple calculation we deduce that $\langle \alpha, \tilde{\beta} \rangle \langle \tilde{\beta}, \alpha \rangle \in \{1, 2, 3\}$. Hence, $\langle \alpha, \tilde{\beta} \rangle > 0$ implies $\langle \alpha, \tilde{\beta} \rangle = 1$. The last claim in the statement follows from the linearity of $\langle \, , \, \rangle$ in the first component. ∎

Let $\Sigma^+ := \{\alpha \in \Phi^+ : \langle \alpha, \tilde{\beta} \rangle = 1\}$. Lemma 7.4.1 allows us to define a fixed-point free involution of Σ^+ defined by $\alpha \mapsto \tilde{\beta} - \alpha$. We pick one element from each equivalence class. Therefore, we have the following disjoint decomposition:

$$\Sigma^+ = \{\alpha_i : 1 \leq i \leq d\} \cup \{\tilde{\beta} - \alpha_i : 1 \leq i \leq d\}. \tag{7.4.2}$$

Hence, $|\Sigma^+| = 2d$, where the value of the integer d is explicitly calculated in Proposition 1.3 of [GW96]. In particular, Table 7.2 is given in [GW96]:

Table 7.2: Values of d

$\mathfrak{g}$		d
A_m	$m \geq 1$	$m-1$
B_m	$m \geq 2$	$2m-3$
C_m	$m \geq 2$	$m-1$
D_m	$m \geq 3$	$2m-4$
G_2		2
F_4		7
E_6		10
E_7		16
E_8		28

Lemma 7.4.2. *The subalgebra $\mathfrak{u}$ is a two-step nilpotent Lie algebra with center $\mathfrak{g}_{\tilde{\beta}}$.*

Proof: It follows from Lemma 7.4.1 that $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{g}_{\tilde{\beta}} \subseteq \mathrm{Z}(\mathfrak{u})$, which implies $\mathfrak{u}$ is a two-step nilpotent Lie algebra. For $\gamma_1, \gamma_2 \in \Sigma^+$, notice that $[\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2}] = 0$ unless $\gamma_2 = \tilde{\beta} - \gamma_1$, and in this case we have $[\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\tilde{\beta}-\gamma_1}] = \mathfrak{g}_{\tilde{\beta}}$. Using these equalities, one can see that $\mathfrak{g}_{\tilde{\beta}} = \mathrm{Z}(\mathfrak{u})$. ∎

Notice that $\mathfrak{u} = \mathfrak{g}_{\tilde{\beta}} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ is of dimension $2d+1$. Let us choose a $(d+1)$-dimensional maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{u}$, defined to be

$$\mathfrak{a} = \mathfrak{g}_{\tilde{\beta}} \oplus \bigoplus_{i=1}^{d} \mathfrak{g}_{\alpha_i}. \tag{7.4.3}$$

The maximality can be seen with the help of Lemma 7.4.1, specifically the fact that $\mathfrak{g}_{\tilde{\beta}} = [\mathfrak{g}_{\tilde{\beta}-\alpha_i}, \mathfrak{g}_{\alpha_i}]$. In Section 7.5, we show that this subalgebra produces a polarizing subgroup of a Heisenberg subgroup.

Lemma 7.4.3. *Let $\alpha \in \Phi$ be an arbitrary root. There exists a simple root $\gamma \in \Delta$ such that $\langle \alpha, \gamma \rangle = \pm 1$ or ± 2.*

Proof: The statement is clear if $\alpha \in \pm\Delta$ (since we can set $\gamma = \pm\alpha$) and so we can assume that $\alpha \in \Phi \setminus \pm\Delta$. In this case there exists a root $\gamma \in \Delta$ such that $\langle \alpha, \gamma \rangle \neq 0$. For root systems other than G_2, the lemma follows from Table 7.1. For G_2, the lemma can be verified by a direct examination of the roots. ∎

Set

$$F(\Phi) := \min\left\{\langle \tilde{\beta}, \alpha \rangle > 0 : \alpha \in \Phi\right\}. \tag{7.4.4}$$

Obviously $F(\Phi) \leq 2$. For the root systems $\mathsf{A}_m, \mathsf{D}_m, \mathsf{E}_6, \mathsf{E}_7$ and E_8 have only one root length and so a similar argument as above shows that $F(\Phi) = 1$ unless $\Phi = \mathsf{A}_1$ which in this case we have $F(\mathsf{A}_1) = 2$. For $\mathsf{B}_m, \mathsf{F}_4$, and G_2, we observe that these root systems have non-perpendicular long roots and so for these root systems we also have $F(\Phi) = 1$.

We show that $F(\mathsf{C}_m) = 2$. If $\langle \tilde{\beta}, \alpha \rangle = 1$ then α is a long root, but in C_m all non-proportional distinct long roots are perpendicular. Hence $F(\mathsf{C}_m) = 2$. Therefore we have

$$F(\Phi) = \begin{cases} 1, & \Phi \neq \mathsf{A}_1; \mathsf{C}_m,\ m \geq 2 \\ 2, & \Phi = \mathsf{A}_1; \mathsf{C}_m,\ m \geq 2. \end{cases} \tag{7.4.5}$$

7.5 Heisenberg subgroups of Chevalley groups

In this section we review the construction of *elementary adjoint Chevalley groups* and we define Heisenberg subgroups of Chevalley groups which are obtained by exponentiating the nilpotent radical of the Heisenberg parabolic subalgebra $\mathfrak{q}$ defined in the previous section. Moreover, we verify that the construction of Chevalley groups over fields given in [Car89, Ste68] can be extended to elementary Chevalley groups defined over $\mathcal{O}/\mathfrak{p}^n$. One way to approach this is to use the language of group schemes [Bor70]. However, in this paper we consider the explicit construction of Chevalley groups using Chevalley bases. The theory of elementary Chevalley groups over rings has also been presented in detail in [VP96].

As before F is a non-Archimedean local field with the ring of integers $\mathcal{O}$, the prime ideal $\mathfrak{p}$ and the residue field $\mathbb{F}_q$, $q = p^l$. Here we assume that $p \geq 3$ and we set $R = \mathcal{O}/\mathfrak{p}^n$, $n \geq 1$. We will use the standard notation, which can be found in [Car89, Ser01, Ste68]. Let

$$\{H_\alpha : \ \alpha \in \Delta\} \cup \{e_\alpha : \ \alpha \in \Phi\}$$

be a *Chevalley basis*, with respect to our choice of base Δ. Let $\mathfrak{g}_\mathbb{Z} \subseteq \mathfrak{g}$ be the free $\mathbb{Z}$-module generated by the Chevalley basis. One can show that $\mathfrak{g}_\mathbb{Z}$ is indeed a Lie algebra over $\mathbb{Z}$. For any $\alpha \in \Phi$ and $\xi \in \mathbb{C}$, $\mathrm{ad}_{\xi e_\alpha} = \xi \mathrm{ad}_{e_\alpha}$ is a nilpotent derivation of $\mathfrak{g}$. Hence, the exponential map

$$x_\alpha(\xi) := \exp(\xi \mathrm{ad}_{e_\alpha})$$

is a Lie algebra automorphism of $\mathfrak{g}$. Moreover, the entries of the matrix of $x_\alpha(\xi)$, with respect to the Chevalley basis, are of the form $a\xi^i$, where $a \in \mathbb{Z}$ and i is a non-negative integer. Let us denote this matrix by $A_\alpha(\xi)$. Consider the R-Lie algebra $\mathfrak{g}_R := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} R$ with the Chevalley basis

$$\{H_\alpha = H_\alpha \otimes 1 : \ \alpha \in \Delta\} \cup \{ \ e_\alpha = e_\alpha \otimes 1 : \ \alpha \in \Phi\}.$$

For every $t \in R$, we obtain a new matrix $\bar{A}_\alpha(t)$ from $A_\alpha(\xi)$, by replacing the entries $a\xi^i$ by $\bar{a}t^i$, where $\bar{a}$ is a reduced modulo $\mathfrak{p}^n$. The linear transformation $\bar{x}_\alpha(t)$ associated with the matrix $\bar{A}_\alpha(t)$ is a Lie algebra automorphism of $\mathfrak{g}_R$. The subgroup of the automorphism group of $\mathfrak{g}_R$, generated by transformations $\bar{x}_\alpha(t)$ for each $\alpha \in \Phi$ and

$t \in R$, is called the *elementary adjoint Chevalley group.* We denote it by $\mathbf{G}_{ad}(R) := \mathbf{G}_{ad}(R, \Phi)$. Let $\alpha \in \Phi$ be an arbitrary root. The *one-parameter subgroup* X_α of $\mathbf{G}_{ad}(R)$ is defined by

$$X_\alpha = \langle \bar{x}_\alpha(t) : t \in R \rangle .$$

Lemma 7.5.1. *The subgroup X_α is isomorphic to the additive group of R.*

Proof: The map $t \to \bar{x}_\alpha(t)$ gives the desired group isomorphism. Note that the injectivity can be seen through the action of $\bar{x}_\alpha(t)$ on the Chevalley basis for the Lie algebra $\mathfrak{g}_R$. More precisely, we have (see [Car89], §4.4) $\bar{x}_\alpha(t) H_\gamma = H_\gamma - \langle \alpha, \gamma \rangle t e_\alpha$. By Lemma 7.4.3, if $\bar{x}_\alpha(t) = \mathbf{1}$, then $t = 0$ since $p \geq 3$. ∎

Let us define the *Heisenberg subgroup* U of $\mathbf{G}_{ad}(R)$

$$U = \left\langle \bar{x}_\alpha(t) : \langle \alpha, \tilde{\beta} \rangle \geq 1,\, t \in R \right\rangle . \tag{7.5.1}$$

Here the right hand side of (7.5.1) is the subgroup of $\mathbf{G}_{ad}(R)$ generated by the given elements $\bar{x}_\alpha(t)$. This subgroup is an analogue of the nilpotent radical of the Heisenberg parabolic subalgebra. This analogy will be apparent in Proposition 7.5.4. From now on, we fix a total ordering $\prec$ of Φ which is compatible with the height function ht, i.e. $\alpha \prec \beta$ implies $\mathrm{ht}(\alpha) \leq \mathrm{ht}(\beta)$. We recall a theorem due to Chevalley (the proof over R is similar to [Car89], Theorem 5.2.2) that expresses the commutator of two generators of $\mathbf{G}_{ad}(R)$ as a product of generators. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$, and let t_1, t_2 be elements of R. Let us define the commutator $[\bar{x}_\alpha(t_2), \bar{x}_\beta(t_1)] := \bar{x}_\alpha(t_2)^{-1} \bar{x}_\beta(t_1)^{-1} \bar{x}_\alpha(t_2) \bar{x}_\beta(t_1)$. The Chevalley commutator formula states that

$$[\bar{x}_\alpha(t_2), \bar{x}_\beta(t_1)] = \prod_{i,j>0} \bar{x}_{i\beta + j\alpha} \left(C_{i,j,\beta,\alpha} (-t_1)^i t_2^j \right) , \tag{7.5.2}$$

where the product is taken over all pairs of positive integers i, j for which $i\beta + j\alpha$ is a root, and the terms of the product are in increasing order of $i + j$. The constants $C_{i,j,\beta,\alpha}$ are in the set $\{\pm 1, \pm 2, \pm 3\}$.

Next, we point out that every element of U can be expressed uniquely in the form

$$\prod_{\langle\alpha,\tilde{\beta}\rangle\geq 1} \bar{x}_\alpha(t_\alpha), \tag{7.5.3}$$

where the product is taken over positive roots α, increasing in the chosen total ordering. Indeed, given an element of U in the form of a product of $\bar{x}_\alpha(t)$'s, the desired order can be achieved by performing a rearrangement as follows: if there is a pair of consecutive terms $\bar{x}_\alpha(t_\alpha)\bar{x}_\beta(t_\beta)$ with $\beta \prec \alpha$, we swap them by use of (7.5.2):

$$\bar{x}_\alpha(t_\alpha)\bar{x}_\beta(t_\beta) = \bar{x}_\beta(t_\beta)\bar{x}_\alpha(t_\alpha)\prod_{i,j>0} \bar{x}_{i\beta+j\alpha}\left(C_{i,j,\beta,\alpha}(-t_\beta)^i t_\alpha^j\right). \tag{7.5.4}$$

In this fashion, $\bar{x}_\beta(t_\beta)\bar{x}_\alpha(t_\alpha)$ is in increasing order, and all the extra terms introduced by the use of the commutator formula are in the desired order because the total ordering $\prec$ is compatible with the height function. This rearrangement terminates after finitely many iterations. The uniqueness of such an expression of elements in U is proved by an argument similar to the proof of [Car89], Theorem 5.3.3(ii). The following lemma can be proved easily.

Lemma 7.5.2. *Let Φ be a root system different from G_2. Then for any α_i, chosen from the decomposition* (7.4.2) *and $t \in R$, we have*

$$\bar{x}_{\alpha_i}(1)\bar{x}_{\tilde{\beta}-\alpha_i}(t) = \bar{x}_{\tilde{\beta}-\alpha_i}(t)\bar{x}_{\alpha_i}(1)\bar{x}_{\tilde{\beta}}(Ct),$$

where $C \in \{\pm 1, \pm 2\}$.

Proof: From (7.5.4) we have

$$\bar{x}_{\alpha_i}(1)\bar{x}_{\tilde{\beta}-\alpha_i}(t) = \bar{x}_{\tilde{\beta}-\alpha_i}(t)\bar{x}_{\alpha_i}(1)\bar{x}_{\tilde{\beta}}(-C_{1,1,\tilde{\beta}-\alpha_i,\alpha_i}t).$$

But (see [Car89], Theorem 5.2.2) $C_{1,1,\tilde{\beta}-\alpha_i,\alpha_i} = \pm(r+1)$, where

$$(\tilde{\beta}-\alpha_i) - r\alpha_i, \cdots, (\tilde{\beta}-\alpha_i), \cdots, (\tilde{\beta}-\alpha_i) + s\alpha_i,$$

is the α_i-chain through $(\tilde{\beta} - \alpha_i)$. Since $\tilde{\beta}$ is the highest root, $(\tilde{\beta} - \alpha_i) + s\alpha_i$ is not a root for $s > 1$, and therefore $s = 1$. Also, it is known that $\langle \tilde{\beta} - \alpha_i, \alpha_i \rangle = r - s$. It follows that

$$r + 1 = \langle \tilde{\beta}, \alpha_i \rangle.$$

Notice that $\langle \tilde{\beta}, \alpha_i \rangle \in \{1, 2\}$, since otherwise $\|\tilde{\beta}\| / \|\alpha_i\| = 3$, which is impossible when the root system is different from G_2. ∎

Lemma 7.5.3. *Let G be a finitely generated group generated by g_i, $1 \leq i \leq n$. Let $A \trianglelefteq G$ and assume that $[g_i, g_j] \in A$ for any $1 \leq i, j \leq n$. Then $[G, G] \subseteq A$.*

Proposition 7.5.4. *Let $p \geq 3$ if G is not of type G_2 and $p \geq 5$ otherwise. Then the group U is two-step nilpotent and $[U, U] = X_{\tilde{\beta}}$.*

Proof: With the help of (7.5.2) and Lemma 7.4.1, one can see that the commutators of the generators of U are in $X_{\tilde{\beta}}$. Hence by applying Lemma 7.5.3 we conclude that the commutator subgroup of U is contained in $X_{\tilde{\beta}}$. Conversely, by Lemma 7.5.2, any element in $X_{\tilde{\beta}}$ can be obtained from commuting suitable elements of X_{α_i} and $X_{\tilde{\beta} - \alpha_i}$, for $1 \leq i \leq d$. Hence, $[U, U] = X_{\tilde{\beta}}$. On the other hand, the fact that $\tilde{\beta}$ is the highest root implies that for every α satisfying $\langle \alpha, \tilde{\beta} \rangle > 0$, we have $i\tilde{\beta} + j\alpha \notin \Phi$ for all $i, j > 0$. Hence, by (7.5.2) we have $[X_\alpha, X_{\tilde{\beta}}] = \mathbf{1}$ which implies that $X_{\tilde{\beta}} \subseteq Z(U)$ and hence U is a two step nilpotent subgroup. ∎

We now recall that by a theorem of Chevalley (whose proof over R is similar to [Car89], Theorem 6.3.1), for any root α there exists a surjective homomorphism

$$\phi_\alpha : \mathrm{SL}_2(R) \longrightarrow \langle X_\alpha, X_{-\alpha} \rangle, \tag{7.5.5}$$

such that

$$\phi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \bar{x}_\alpha(t), \qquad \phi_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \bar{x}_{-\alpha}(t).$$

For any invertible element $\lambda \in R$, we denote

$$h_\alpha(\lambda) := \phi_\alpha \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{7.5.6}$$

Let $\alpha, \beta \in \Phi$ be any roots, then one can show that (see [Car89], Chapter 7)

$$h_\alpha(\lambda)\bar{x}_\beta(t)h_\alpha(\lambda)^{-1} = \bar{x}_\beta\left(\lambda^{\langle\beta,\alpha\rangle}t\right). \tag{7.5.7}$$

7.6 Faithful representations and generic characters

Let (ρ, V) be a faithful representation of $\mathbf{G}_{ad}(R)$. Let $\sigma := \rho|_U$, be the restriction of ρ to the Heisenberg subgroup U, defined in (7.5.1), and let (σ_i, V_i), $1 \leq i \leq k$, be the irreducible factors in the decomposition of the U-representation (σ, V). Then by Schur's lemma, for any $z \in Z(U)$ and $v \in V_i$, we have $\sigma_i(z)v = \chi_i(z)v$, where χ_i is a one-dimensional representation of $Z(U)$. By Lemma 7.2.1 for each $1 \leq i \leq k$ there exists $\bar{b}_i = b_i + \mathfrak{p}^{-\ell} \in \mathfrak{p}^{-(n+\ell)}/\mathfrak{p}^{-\ell}$ such that for any $s \in \mathcal{O}$,

$$\chi_i\left(\bar{x}_{\tilde{\beta}}(s + \mathfrak{p}^n)\right) = \psi(\mathrm{Tr}(b_i s)). \tag{7.6.1}$$

With this observation we prove the following proposition. As before the characteristic of the residue field $\mathcal{O}/\mathfrak{p}$ is p.

Proposition 7.6.1. *Let $p \geq 3$ when $\Phi \neq \mathsf{G}_2$ and $p \geq 5$ when $\Phi = \mathsf{G}_2$. Let χ_i, $1 \leq i \leq k$ be defined as above, and let $b_i \in \mathfrak{p}^{-(n+\ell)}$ correspond to χ_i by Lemma 7.2.1. Then $\nu(b_i) = -(n+\ell)$ for some $1 \leq i \leq k$. In particular, χ_i is a generic character of $Z(U)$.*

Proof: Suppose that for each $1 \leq i \leq k$ we have $\varpi^{n-1} b_i \in \mathfrak{p}^{-\ell}$. Then $\chi_i(\bar{x}_{\tilde{\beta}}(\varpi^{n-1} + \mathfrak{p}^n)) = 1$. This in particular implies that $\rho(\bar{x}_{\tilde{\beta}}(\varpi^{n-1} + \mathfrak{p}^n)) = \mathbf{1}$ which is a contradiction since ρ is assumed to be a faithful representation. This proves the existence of $1 \leq i \leq k$ such that $\nu(b_i) = -(n+\ell)$.

Next we prove that if $\nu(b_i) = -(n+\ell)$ then χ_i is a generic character of $Z(U)$. For $u \in U$ let $\chi_i([u,y]) = 1$ for all $y \in U$. By (7.5.3),

$$u = \prod_{\langle \alpha, \tilde{\beta} \rangle \geq 1} \bar{x}_\alpha(s_\alpha + \mathfrak{p}^n) \qquad s_\alpha \in \mathcal{O}, \tag{7.6.2}$$

where the product is taken over positive roots α, increasing in the chosen total ordering. We will show that the only term that contributes to (7.6.2) is the term that belongs to $X_{\tilde{\beta}}$. It follows that $u \in Z(U)$. Note that, for any $x \in U$, the map $y \mapsto [x,y]$ is a group homomorphism, since U is a two-step nilpotent group.

We remark that for $\alpha, \beta \in \Sigma^+$ we have $[X_\alpha, X_\beta] = \mathbf{1}$ unless $\beta = \tilde{\beta} - \alpha$. For any $\alpha \neq \tilde{\beta}$ in (7.6.2) and arbitrary $s \in \mathcal{O}$, from the Chevalley commutator formula (7.5.2) we have

$$[u, \bar{x}_{\tilde{\beta}-\alpha}(s + \mathfrak{p}^n)] = \bar{x}_{\tilde{\beta}}(C s_\alpha s + \mathfrak{p}^n),$$

where by Lemma 7.5.2, $C \in \{\pm 1, \pm 2\}$ if Φ is different from G_2 and $C \in \{\pm 1, \pm 2, \pm 3\}$ when $\Phi = \mathsf{G}_2$. Since for any $s \in \mathcal{O}$ we have

$$1 = \chi_i\left([u, \bar{x}_{\tilde{\beta}-\alpha}(s + \mathfrak{p}^n)]\right) = \chi_i\left(\bar{x}_{\tilde{\beta}}(C s_\alpha s + \mathfrak{p}^n)\right) = \psi(\mathrm{Tr}(C b_i s_\alpha s)), \tag{7.6.3}$$

then

$$C b_i s_\alpha \in \mathfrak{p}^{-\ell},$$

which implies that $\nu(s_\alpha) \geq n$ since $\nu(b_i) = -(n+\ell)$ (when $\Phi = \mathsf{G}_2$ we must assume $p \geq 5$ since C can be ± 3). Hence $s_\alpha \in \mathfrak{p}^n$. This shows that $u = \bar{x}_{\tilde{\beta}}(s_{\tilde{\beta}} + \mathfrak{p}^n)$ for some $s_{\tilde{\beta}} \in \mathcal{O}$ and so $u \in X_{\tilde{\beta}} \subseteq Z(U)$ which shows that χ_i is a generic character of $Z(U)$. ∎

In order to apply the Stone-von Neumann theorem, we then need to find the polarizing subgroup of U. Define

$$A = \left\langle \bar{x}_{\tilde{\beta}}(t),\ \bar{x}_{\alpha_i}(t) : 1 \leq i \leq d,\, t \in R \right\rangle, \tag{7.6.4}$$

where the α_i are chosen with respect to the decomposition (7.4.2). We will show that A is a polarizing subgroup of U with respect to the generic character χ_i defined above. Notice that for any α_i and α_j, $1 \leq i, j \leq d$, chosen from the first disjoint component of the decomposition (7.4.2), neither $\alpha_i + \alpha_j$ nor $\alpha_i + \tilde{\beta}$ is a root. Hence, the right hand side of (7.5.2) is always zero for elements in A, and therefore A is an abelian subgroup of U containing $X_{\tilde{\beta}}$.

Proposition 7.6.2. *Let $p \geq 3$ when $\Phi \neq \mathsf{G}_2$ and $p \geq 5$ when $\Phi = \mathsf{G}_2$. Let χ_1 be a one-dimensional representation of $Z(U)$ corresponding to $b_1 \in \mathfrak{p}^{-(n+\ell)}$ via Lemma 7.2.1. Assume that $\nu(b_1) = -(n+\ell)$. Then A is a polarizing subgroup with respect to χ_1.*

Proof: We have shown that A is an abelian subgroup and so A is an isotropic subgroup of U. Assume A is not a polarizing subgroup. Each $u \in U$ has a unique presentation (7.6.2). By the length of $u \in U$ we mean the number of terms in (7.6.2). Let $u \in U \setminus A$ be an element with the shortest length such that

$$\chi_1\left([u, a]\right) = 1, \qquad \forall a \in A. \tag{7.6.5}$$

Let us denote the unique presentation of u as follows

$$u = \prod_{\langle \alpha, \tilde{\beta} \rangle \geq 1} \bar{x}_\alpha(s_\alpha + \mathfrak{p}^n) \qquad s_\alpha \in \mathcal{O}. \tag{7.6.6}$$

We claim that the leftmost term in the product in (7.6.6) cannot belong to either of $X_{\tilde{\beta}}$ and X_{α_i}, $1 \leq i \leq d$. That is because otherwise, one can eliminate this term and obtain another element in $U \setminus A$ with shorter length that satisfies (7.6.5). We again remark that for $\alpha, \beta \in \Sigma^+$ we have $[X_\alpha, X_\beta] = \mathbf{1}$ unless $\beta = \tilde{\beta} - \alpha$. Without loss of generality we can write

$$u = \bar{x}_{\tilde{\beta} - \alpha_1}(s_{\alpha_1} + \mathfrak{p}^n) u' \qquad s_{\alpha_1} \in \mathcal{O} \setminus \mathfrak{p}^n,$$

where α_1 is taken from the decomposition (7.4.2). Notice that none of the elements of $X_{\tilde{\beta}-\alpha_1}$ appear in the factorization of u'. Hence for an arbitrary $s \in \mathcal{O}$ we have

$$\begin{aligned}[u, \bar{x}_{\alpha_1}(s+\mathfrak{p}^n)] &= [\bar{x}_{\tilde{\beta}-\alpha_1}(s_{\alpha_1}+\mathfrak{p}^n), \bar{x}_{\alpha_1}(s+\mathfrak{p}^n)][u', \bar{x}_{\alpha_1}(s+\mathfrak{p}^n)] \\ &= [\bar{x}_{\tilde{\beta}-\alpha_1}(s_{\alpha_1}+\mathfrak{p}^n), \bar{x}_{\alpha_1}(s+\mathfrak{p}^n)] = \bar{x}_{\tilde{\beta}}(Cs_{\alpha_1}s+\mathfrak{p}^n)\end{aligned}$$

where $C \in \{\pm 1, \pm 2, \pm 3\}$. Hence from (7.6.5) we deduce that for any $s \in \mathcal{O}$

$$1 = \chi_1\left([u, \bar{x}_{\alpha_1}(s+\mathfrak{p}^n)]\right) = \psi(\mathrm{Tr}(Cb_1 s_{\alpha_1} s)). \tag{7.6.7}$$

Therefore we should have $Cb_1 s_{\alpha_1} \in \mathfrak{p}^{-\ell}$. By Lemma 7.5.2, for all root systems other than G_2, the contradiction $s_{\alpha_1} \in \mathfrak{p}^n$ is obtained when $p \geq 3$. However, for the root system G_2 the further assumption $p \geq 5$ is required in order to obtain a contradiction. ∎

We now compute the index of A in U.

Lemma 7.6.3. *Let d be as in Table 7.2. Then $[U : A] = q^{nd}$.*

Proof: Recall that $|\Sigma^+| = 2d$. From (7.5.1) and the uniqueness of the product in (7.5.3) we deduce that $|U| = q^{(2d+1)n}$ and $|A| = q^{(d+1)n}$. Therefore, $[U : A] = q^{nd}$. ∎

7.7 Proof of Theorem 7.1.1

In what follows the group of units of a given ring R is denoted by $R^\times$. Let (ρ, V) be a faithful representation of $\mathbf{G}_{ad}(R)$. Let $\sigma := \rho|_U$ be the restriction of ρ to the Heisenberg subgroup U, and let (σ_i, V_i), $1 \leq i \leq k$, be the irreducible factors in the decomposition of the U-representation (σ, V) with central characters χ_i. By Proposition 7.6.1, we can assume that χ_1 is a generic character of $Z(U)$, such that the element $b_1 \in \mathfrak{p}^{-(n+\ell)}$ associated to χ_1 in Lemma 7.2.1 satisfies $\nu(b_1) = -(n+\ell)$.

Then, by Proposition 7.6.2, A is a polarizing subgroup with respect to the generic character χ_1. Therefore, by the Stone-von Neumann theorem and Lemma 7.6.3, we have

$$\dim(V_1) = [U : A] = q^{dn}.$$

For any $\bar{\lambda} = \lambda + \mathfrak{p}^n \in R^\times$, and any root $\alpha \in \Phi$, consider the element $h_\alpha(\bar{\lambda})$ introduced in (7.5.6). It follows from (7.5.7) that $h_\alpha(\bar{\lambda})$ normalizes any one-parameter subgroup. Therefore, $h_\alpha(\bar{\lambda})$ normalizes U. Define the U-representation $\sigma^{\bar{\lambda},\alpha}$ to be the conjugation of σ by $h_\alpha(\bar{\lambda})$:

$$\sigma^{\bar{\lambda},\alpha} : U \to \mathrm{GL}(V), \qquad u \mapsto \sigma\left(h_\alpha(\bar{\lambda}) u h_\alpha(\bar{\lambda})^{-1}\right).$$

Notice that the U-intertwining operator $\rho(h_\alpha(\bar{\lambda}))$ gives a U-isomorphism between (σ, V) and $(\sigma^{\bar{\lambda},\alpha}, V)$. Therefore, $(\sigma_1^{\bar{\lambda},\alpha}, V_1)$ is also an irreducible subrepresentation of (σ, V). Hence for any $z \in X_{\tilde{\beta}} \subseteq Z(U)$ and $v \in V_1$ we have

$$\chi_1(h_\alpha(\bar{\lambda}) z h_\alpha(\bar{\lambda})^{-1}) v = \sigma_1(h_\alpha(\bar{\lambda}) z h_\alpha(\bar{\lambda})^{-1}) v = \sigma_1^{\bar{\lambda},\alpha}(z) v = \chi_1^{\bar{\lambda},\alpha}(z) v,$$

where $\chi_1^{\bar{\lambda},\alpha}$ is the one-dimensional representation of $Z(U)$ which is the central character of $\sigma_1^{\bar{\lambda},\alpha}$. Then for $s + \mathfrak{p}^n \in R \cong X_{\tilde{\beta}}$, from (7.5.7), we obtain

$$\chi_1^{\bar{\lambda},\alpha}\left(\bar{x}_{\tilde{\beta}}(s + \mathfrak{p}^n)\right) = \chi_1\left(\bar{x}_{\tilde{\beta}}(\lambda^{\langle\tilde{\beta},\alpha\rangle} s + \mathfrak{p}^n)\right) = \psi\left(\mathrm{Tr}\left(b_1 \lambda^{\langle\tilde{\beta},\alpha\rangle} s\right)\right).$$

Next, we count the number of mutually distinct one-dimensional representations of $\chi_1^{\bar{\lambda},\alpha}$. We consider two cases:

1. The root system Φ is not A_1 or C_m for $m \geq 2$: Equation (7.4.5) implies that there exists a root $\alpha \in \Phi$ such that $\langle\tilde{\beta}, \alpha\rangle = 1$. Hence, for this particular root α we have

$$\chi_1^{\bar{\lambda},\alpha}(\bar{x}_{\tilde{\beta}}(s + \mathfrak{p}^n)) = \chi_1(\lambda \bar{x}_{\tilde{\beta}}(s + \mathfrak{p}^n)) = \psi\left(\mathrm{Tr}\left(b_1 \lambda s\right)\right), \qquad \forall s \in \mathcal{O}.$$

Since χ_1 corresponds to $b_1 \in \mathfrak{p}^{-(n+\ell)}$ with $\nu(b_1) = -(n+\ell)$ then we can conclude that $\chi_1^{\bar{\lambda}_1,\alpha} \neq \chi_1^{\bar{\lambda}_2,\alpha}$ when $\bar{\lambda}_1 \neq \bar{\lambda}_2 \in R^\times$, since otherwise $b_1(\lambda_1 - \lambda_2) \in \mathfrak{p}^{-\ell}$ which implies $\lambda_1 - \lambda_2 \in \mathfrak{p}^n$. Hence, there are $q^n - q^{n-1}$ distinct one-dimensional representations $\chi_1^{\bar{\lambda},\alpha}$ and therefore, there are at least $q^n - q^{n-1}$ non-isomorphic irreducible subrepresentations of V, each of dimension q^{dn}. Hence,

$$\dim(V) \geq (q^n - q^{n-1})q^{dn},$$

where d is given in Table 7.2.

2. The root system Φ is either C_m or A_1: Equation (7.4.5) implies that for no root in Φ, $\langle\tilde{\beta}, \alpha\rangle = 1$; but a root α can be chosen such that $\langle\tilde{\beta}, \alpha\rangle = 2$. Then for $s + \mathfrak{p}^n \in R \cong X_{\tilde{\beta}}$ we have

$$\chi_1^{\bar{\lambda},\alpha}(\bar{x}_{\tilde{\beta}}(s + \mathfrak{p}^n)) = \chi_1\left(\bar{x}_{\tilde{\beta}}(\lambda^2 s + \mathfrak{p}^n)\right) = \psi\left(\mathrm{Tr}\left(b_1\lambda^2 s\right)\right).$$

Hence, we can construct $|R^\times|/2 = (q^n - q^{n-1})/2$ distinct one-dimensional representations $\chi_1^{\bar{\lambda},\alpha}$, which leads to obtaining $(q^n - q^{n-1})/2$ non-isomorphic factors in the decomposition of (σ, V), each of dimension $q^{(m-1)n}$. Hence

$$\dim(V) \geq \frac{1}{2}(q^n - q^{n-1})q^{(m-1)n}.$$

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