The Refined Solution to the Capelli Eigenvalue Problem for $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and $\mathfrak{gl}(m|2n)$

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Abstract

In this thesis, we consider the question of describing the eigenvalues of a distinguished family of invariant differential operators associated to a Lie superalgebra $\mathfrak{g}$ and a $\mathfrak{g}$-module W, called the "Capelli basis", via evaluation of certain classes of supersymmetric functions, called the interpolation super Jack polynomials. Finding the eigenvalues of the Capelli basis is referred to the *Capelli Eigenvalue Problem.* The eigenvalue formula depends on the chosen parametrization of the highest weight vectors in the decomposition of the superpolynomial algebra $\mathcal{P}(W)$, and consequently on the choice of a Borel subalgebra. In this thesis, we give a solution for each conjugacy class of Borel subalgebras, which we call a refined solution to the Capelli Eigenvalue Problem.

Given the pair $(\mathfrak{g}, W)$, we investigate the formulae for the eigenvalues of the Capelli operators associated to the completely reducible and multiplicity-free modules $\left(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)$ and $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$. In the former case, we show that we can express the eigenvalue of the Capelli operator on the irreducible component W_λ of the multiplicity-free decomposition of $\mathcal{P}(W)$ as a polynomial function of the $\mathfrak{b}$-highest weight of W_λ for any Borel subalgebra $\mathfrak{b}$.

In the latter case, we show with a concrete counterexample that we cannot expect the results to be as strong as in the first case for all Borel subalgebras. We then express the eigenvalue of the Capelli operator on the irreducible component W_λ of the multiplicity-free decomposition of $\mathcal{P}(W)$ as a polynomial function of $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$, where $\tau_{\mathfrak{b}}$ is a *piecewise* affine map on the span of $\mathfrak{b}$-highest weights of the irreducible submodules of $\mathcal{P}(W)$, with respect to different decreasing Borel subalgebras $\mathfrak{b}$.

Dedications

to the ones that I love and the ones who helped, supported and encouraged me.

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Per aspera ad astra

Lucius Annaeus Seneca

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Chapter 1

Introduction

Let $\mathfrak{g}$ be a Lie superalgebra. Let W be a $\mathfrak{g}$-module such that the superpolynomial algebra $\mathcal{P}(W)$ is completely reducible and has a multiplicity-free decomposition which is parametrized by a set Ω, that is,

$$\mathcal{P}(W) \cong \bigoplus_{\lambda \in \Omega} W_\lambda,$$

where the W_λ's are pairwise non-isomorphic irreducible $\mathfrak{g}$-modules. Then, Sahi showed that the algebra of $\mathfrak{g}$-invariant polynomial-coefficient differential operators, denoted $\mathcal{PD}(W)$, admits a natural basis D^μ where $\mu \in \Omega$ (see [Sah94]). By Schur's Lemma, each D^μ acts by a scalar on the irreducible module W_λ and finding the eigenvalue of D^μ on W_λ for $\mu, \lambda \in \Omega$ is referred to as the *Capelli Eigenvalue Problem* (abbreviated as CEP).

Finding the eigenvalues of D^μ on W_λ has a long history. In 1991, Kostant and Sahi constructed examples of Capelli-like operators associated to Jordan algebras in [KS91]. In the early 1990's, Sahi considered in [Sah94] the Capelli basis associated to a Hermitian symmetric pair and proved that the eigenvalues of such operators are polynomials that are characterized by certain symmetry, vanishing and degree constraints. In [KS96], Knop and Sahi found the connections between Sahi's interpolation polynomials and the Jack polynomials. Sahi's interpolation polynomials were later studied by Okounkov and Olshanski in [OO97], who referred to them as shifted Jack polynomials. In 2020, Sahi, Salmasian and Serganova extended the notion of Capelli operators to Lie superalgebras in [SSS20]. They described a family of multiplicity-free representations that correspond to Jordan superalgebras. They computed the eigenvalues of the resulting basis of invariant differential operators and gave a formula in terms of Sergeev-Veselov polynomials, as defined in[SV05].

In particular, in [SSS20], Sahi, Salmasian and Serganova found a solution to the CEP for Capelli operators associated to $\left(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)$ and $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$. A priori their solution depends on the partition parametrization of the highest weights and is specific to the choice of the Borel subalgebra. Our goal in this thesis is to prove a formula for this eigenvalue as a function of the $\mathfrak{b}$-highest weight of W_λ, for any Borel subalgebra $\mathfrak{b}$. That is, we provide a formula for this eigenvalue that disregards this partition parametrization that lies in the background.

The outline of the thesis is as follows. In Chapter 2, we give an introduction to the structure theory of the so-called basic classical Lie superalgebras $\mathfrak{g}$ and the highest weight theory of $\mathfrak{g}$. We also introduce the Weyl vectors with respect to different Borel subalgebras of $\mathfrak{g}$. In Chapter 3, we study several algebras associated to W, namely, supersymmetric algebras $\mathcal{S}(W)$, $\mathcal{P}(W)$, $\mathcal{PD}(W)$, and $\mathcal{D}(W)$, the constant-coefficient differential operators on W. We also define carefully the Capelli Eigenvalue Problem. In Chapter 4, we give an introduction to supersymmetric polynomials, and in particular, give precise definitions of Jack polynomials, interpolation Jack polynomials and interpolation super Jack polynomials.

Our new results start in Chapter 5. In Chapter 5, we find a refined solution to the CEP for $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, $W = \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*$ with respect to arbitrary Borel subalgebras by using Weyl vectors defined in Chapter 2.

In Chapter 6, we first give the explicit form of the solution to the CEP for $\mathfrak{g} = \mathfrak{gl}(m|2n)$, $W = \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)$ introduced in [SSS20], which depends on the parametrization of the highest weights with respect to $\mathfrak{b}_{\mathrm{op}}$, the opposite standard Borel subalgebra of $\mathfrak{gl}(m|2n)$. Then we define decreasing Borel subalgebras. For each decreasing Borel subalgebra $\mathfrak{b}$, we give an explicit formula for the corresponding highest weight $\underline{\lambda}_{\mathfrak{b}}$ of the irreducible component W_λ of the multiplicity-free decomposition of $\mathcal{P}(W)$. We then define compatible and incompatible highest weights with respect to decreasing Borel subalgebras. We show that the eigenvalue of D^μ on W_λ can be expressed as a polynomial function composed with a piecewise affine map evaluated on $\underline{\lambda}_{\mathfrak{b}}$.

Our main original contributions in this thesis are as follows.

In the setting of $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and $W = \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*$, let $\mathfrak{h} = \mathfrak{h}_{m|n} \oplus \mathfrak{h}_{m|n}$ be the standard Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}' \oplus \mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. We define a map $\mathfrak{h}_{m|n} \to \mathbb{C}^{m|n}$ such that $\alpha \mapsto \tilde{\alpha}$ in Remark 5.2.1. We then prove the following theorem.

Theorem (Theorem 5.2.3 and Corollary 5.2.5)**.** *For any $(\eta, \eta') \in \mathfrak{h}^*$, define an affine map $\tau_{\mathfrak{b}} : \mathfrak{h}^* \to \mathbb{C}^{m|n}$ by*

$$\tau_{\mathfrak{b}}((\eta, \eta')) \coloneqq \widetilde{\eta}' + \widetilde{\rho}_{\mathfrak{b}}.$$

The eigenvalue of the Capelli operator D^μ on the submodule W_λ with highest weight $(\underline{\lambda}_{\mathfrak{b}'}, \underline{\lambda}_{\mathfrak{b}})$ with respect to $\mathfrak{b}' \oplus \mathfrak{b}$ is equal to

$$SP^*_{\mu,1} \circ \tau_{\mathfrak{b}}\left((\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})\right),$$

where $SP^*_{\mu,1}$ is the interpolation super Jack Polynomial from Remark 4.4.5.

In the setting of $\mathfrak{g} = \mathfrak{gl}(m|2n)$ and $W = \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)$, we introduce the concepts of very even Borel subalgebras, relatively even Borel subalgebras, generic and nongeneric highest weights, compatible and incompatible highest weights with respect to decreasing Borel subalgebras, which facilitate the understanding of representation theory of $\mathfrak{gl}(m|2n)$ and the CEP for $\mathfrak{gl}(m|2n)$.

Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be a decreasing Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. We define a set $\mathcal{C}_{\mathfrak{b}}$ of matrices of size $(m+n)\times(m+2n)$ and a vector $X_{\mathfrak{b}}$ for each decreasing Borel subalgera of $\mathfrak{g}$ in Definition 6.5.5 and Equation (6.5.3) respectively. Then in the case of λ being compatible with $\mathfrak{b}$, we prove our refined solution to the CEP is as follows.

Theorem (Proposition 6.5.17)**.** *Define $\tau_{\mathfrak{b}} : \mathfrak{h}^* \to \mathbb{C}^{m|n}$ by*

$$\tau_{\mathfrak{b}}(\eta) = M_{\mathfrak{b}}\eta + X_{\mathfrak{b}}.$$

If λ is compatible with $\mathfrak{b}$, then the eigenvalue of the Capelli operator D^{μ} on the submodule W_λ with highest weight $\underline{\lambda}_{\mathfrak{b}}$ is

$$SP^*_{\mu,\frac{1}{2}} \circ \left(\tau_{\mathfrak{b}}\left(\underline{\lambda}_{\mathfrak{b}}\right)\right)$$

where $SP^*_{\mu,\frac{1}{2}}$ is the interpolation super Jack polynomial from Remark 4.4.5.

We then give a surprising example in $\mathfrak{gl}(2|2)$ to illustrate that in other cases, we *cannot* find affine maps so that a refined solution to the CEP can be found.

We then go on to show that although there is not one single affine map that serves our purposes, we can define $\tau_{\mathfrak{b}}$ as a piecewise affine map. If λ is incompatible with $\mathfrak{b}$, we define a unique value $I_{\lambda,\mathfrak{b}}$ and a set $\mathcal{C}_{b,I_{\lambda,\mathfrak{b}}}$ of matrices of size $(m+n)\times(m+2n)$.

For further clarity, we give the culmination of our results, which includes the previous theorem as a special case, as a single theorem. Then the refined solution to the CEP is given as follows.

Theorem. *Retain the set-up as above. If*

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = \begin{cases} M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} & \text{for any } M_{\mathfrak{b}} \in \mathcal{C}_b, \text{ if } \lambda \text{ is compatible with } \mathfrak{b} \\ M_{b,I_{\lambda,\mathfrak{b}}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} & \text{for any } M_{b,I_{\lambda,\mathfrak{b}}} \in \mathcal{C}_{b,I_{\lambda,\mathfrak{b}}}, \text{ if } \lambda \text{ is incompatible with } \mathfrak{b}, \end{cases}$$

then the eigenvalue of the Capelli operator D^μ on the submodule W_λ is equal to

$$SP^*_{\mu,\frac{1}{2}}\left(\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})\right),$$

where $SP^*_{\mu,\frac{1}{2}}$ is the interpolation super Jack Polynomial from Remark 4.4.5.

There are also some minor original contributions in this thesis. We list them as follows.

(i) We introduce the concept of increasing (decreasing) Borel subalgebras to understand the structure theory of Lie superalgebras and solve the Capelli Eigenvalue Problem in its refined version.

(ii) For a $\mathfrak{g}$-module (π, W), we give two explicit $\mathfrak{g}$-module structures of $\mathcal{D}(W)$, denoted $(\pi_{\mathcal{D}}, \mathcal{D}(W))$ and $(\pi_{\tilde{\mathcal{D}}}, \mathcal{D}(W))$. We then show in Section 3.3 that these $\mathfrak{g}$ actions on $\mathcal{D}(W)$ are the same.

(iii) For a decreasing Borel subalgebra $\mathfrak{b}$ of $\mathfrak{gl}(m|2n)$, we introduce two sets of indices $\ell_{i,\mathfrak{b}}$ and $j_{k,\mathfrak{b}}$. We then in Section 6.4.1 show that $\mathfrak{b}$ in completely determined by either of these sets of indices.

(iv) For each irreducible submodule of the $\mathfrak{gl}(m|2n)$-module $\mathcal{P}\left(\mathcal{S}^2\left(\mathbb{C}^{m|n}\right)\right)$ and for each decreasing Borel subalgebra of $\mathfrak{gl}(m|2n)$, we provide an explicit formula in Section 6.4.2 for the highest weight in terms of its highest weight with respect to the opposite standard Borel subalgebra.

In future work, it would be interesting to consider the remaining cases in [SSS20], to see what kinds of solutions their refined versions admit. It would also be interesting to establish a uniqueness result for the solution to the $\mathfrak{gl}(m|2n)$ case solved here, although we conjecture that the form of our solution is the simplest possible. At the same time, we made a choice of $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$ in Section 6.4 in order to proceed and have seen concretely in Section 6.3 that other choices may be possible. Also, it is interesting to understand the reason why the eigenvalues cannot be expressed as a polynomial function evaluating at a single affine map of highest weights for the not relatively even Borel subalgebras of $\mathfrak{gl}(m|2n)$.

Chapter 2

On Lie superalgebras and their modules

Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the additive group of two elements. A complex finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called *classical* if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a semisimple $\mathfrak{g}_{\bar{0}}$-module. A classical Lie superalgebra is called *basic* if it admits an even nondegenerate invariant bilinear form. A classical Lie superalgebra which is not basic is called *strange*. For example, the general Lie superalgebra $\mathfrak{gl}(m|n)$ is basic classical, the queer Lie superalgebra $\mathfrak{q}(n)$ is classical strange, and the Cartan type Lie superalgebra $W(n)$ is not classical. In this chapter, let $\mathfrak{g}$ be a basic classical Lie superalgebra. We discuss the structure theory of $\mathfrak{g}$ in Section 2.1, and the highest weight theory of $\mathfrak{g}$ in Section 2.2.

In this thesis, we focus on basic classical Lie superalgebras, we refer the reader to [Kac77] for a complete list of *simple* Lie superalgebras of any type.

2.1 Structure Theory of basic classical Lie Superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) \coloneqq \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$. When $\mathfrak{g}$ is finite-dimensional basic classical, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincides with a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$, and in turn, all the Cartan subalgebras are conjugate by the Weyl group of $\mathfrak{g}_{\bar{0}}$. In particular, unless otherwise stated, we assume $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{g}$ is a *simple* basic classical Lie superalgebra. We first discuss the root system of $\mathfrak{g}$ and, introduce odd and even reflections. Unlike the case of simple Lie algebras, Borel subalgebras are not all conjugate under the Weyl group. We introduce $\epsilon\delta$-sequences as tools to classify the Weyl group conjugacy classes of Borel subalgebras.

2.1.1 Root systems

In this subsection we define the root systems of $\mathfrak{g}$. Then we compute a root system for $\mathfrak{gl}(m|n)$ explicitly.

Definition 2.1.1. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. For $\alpha \in \mathfrak{h}^*$, define*

$$\mathfrak{g}_\alpha \coloneqq \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \text{ for all } h \in \mathfrak{h}\}.$$

The root system *for $\mathfrak{g}$ with respect to $\mathfrak{h}$ is*

$$\Phi \coloneqq \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}.$$

Moreover, we define the set of even *and* odd *roots to be*

$$\Phi_{\bar{0}} = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0\} \qquad \textit{and} \qquad \Phi_{\bar{1}} = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0\}$$

respectively.

For a root system Φ, we can define the set of positive, negative and simple roots as follows. For any vector space, given a linear functional $f : V \to \mathbb{R}$, we can define a total ordering on V by $a \succ b$ if and only if $f(a - b) > 0$ for all $a, b \in V$.

Definition 2.1.2. *Let Φ be a root system of $\mathfrak{g}$. A positive system Φ^+ is a subset of Φ consisting of roots $\alpha \in \Phi$ such that $\alpha \succ 0$ with respect to the total ordering $\succ$ on the vector space spanned by Φ. Fix a positive system Φ^+. We define*

1. *the negative system by $\Phi^- := -\Phi^+$, and*
2. *$\Pi \subseteq \Phi^+$ to be the set of $\alpha \in \Phi^+$ such that α cannot be written as a sum of two roots in Φ^+. Then Π is called the simple system corresponding to Φ^+.*

Remark 2.1.3. Similar to classical Lie algebras, the simple systems and positive systems of $\mathfrak{g}$ are in one-to-one correspondence.

We finish this subsection by giving a matrix realization of $\mathfrak{gl}(m|n)$ and in turn, give an example of positive system of $\mathfrak{gl}(m|n)$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded super vector space. Then $\mathrm{End}(V)$ equipped with the standard Lie superbracket is a Lie superalgebra called the general linear Lie superalgebra, denoted by $\mathfrak{gl}(V)$ or $\mathfrak{gl}(m|n)$ if $V = \mathbb{C}^{m|n}$. Let $\{e_1, \dots, e_m\}$ and $\{e_{m+1}, \dots, e_{m+n}\}$ be ordered bases for $V_{\bar{0}}$ and $V_{\bar{1}}$ respectively so that their union is a homogeneous ordered basis for V. Then any element $X \in \mathfrak{gl}(m|n)$ can be written as an $(m + n) \times (m + n)$ complex matrix of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1.1}$$

where A, B, C and D are $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices respectively.

Moreover, as a Lie superalgebra, the even part of $\mathfrak{gl}(m|n)$ is

$$\mathfrak{gl}(m|n)_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$$

which consists of block matrices of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and the odd part $\mathfrak{gl}(m|n)_{\bar{1}}$ consists of block matrices of the form $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{gl}(m|n)_{\bar{0}}$. That is, $\mathfrak{h}$ has a basis $\{E_{i,i}, E_{m+j,m+j}\}_{1\leq i\leq m, 1\leq j\leq n}$. Let $\{\epsilon_i, \delta_j\}_{i,j}$ be the basis of $\mathfrak{h}^*$ dual to $\{E_{i,i}, E_{m+j,m+j}\}_{i,j}$. The standard root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ is given by

$$\Phi_{\bar{0}} = \{\pm(\epsilon_i - \epsilon_j), \pm(\delta_i - \delta_\ell) \mid 1 \leq i \neq j \leq m, 1 \leq k \neq \ell \leq n\}, \tag{2.1.2}$$

and

$$\Phi_{\bar{1}} = \{\pm(\epsilon_i - \delta_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}. \tag{2.1.3}$$

The standard simple system for $\mathfrak{gl}(m|n)$ is

$$\Pi = \{\delta_i - \delta_{i+1}, \delta_m - \epsilon_1, \epsilon_j - \epsilon_{j+1} \mid 1 \leq i \leq m-1, 1 \leq j \leq n-1\}.$$

2.1.2 The Supertrace and Non-degenerate Bilinear Forms

In this subsection, we define the supertrace of $\mathfrak{gl}(m|n)$, which gives rise to an even supersymmetric nondegenerate bilinear form on $\mathfrak{gl}(m|n)$.

Definition 2.1.4. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space. We say a bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{C}$ is*

1. even *if $(V_i, V_j) = 0$ whenever $i + j = \bar{1}$ for all $i, j \in \mathbb{Z}_2$, and*
2. supersymmetric *if $(\cdot,\cdot)_{V_{\bar{0}} \times V_{\bar{0}}}$ is symmetric and $(\cdot,\cdot)_{V_{\bar{1}} \times V_{\bar{1}}}$ is anti-symmetric*

When $X \in \mathfrak{gl}(m|n)$ is in the block matrix form given in Equation (2.1.1), we define the supertrace of X as

$$\mathrm{str}(X) \coloneqq \mathrm{tr}(A) - \mathrm{tr}(D) \tag{2.1.4}$$

where tr denotes the usual trace of a matrix. The supertrace naturally defines the following bilinear form on $\mathfrak{gl}(m|n)$:

$$(\cdot,\cdot) : \mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n) \to \mathbb{C} \text{ such that } (X, Y) \mapsto \mathrm{str}(XY), \tag{2.1.5}$$

where juxtaposition denotes the usual matrix product. Notice that the bilinear form defined in Equation (2.1.5) is even supersymmetric. Also notice that $\mathfrak{gl}(m|n)_{\bar{0}}$ and $\mathfrak{gl}(m|n)_{\bar{1}}$ are orthogonal with respect to $(\cdot,\cdot)$. This bilinear form restricts to a nondegenerate bilinear form on $\mathfrak{gl}(m|n)_{\bar{0}}$, which we recognize as the standard Killing form on $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

Now we define a bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^*$, also denoted as $(\cdot,\cdot)$. Let $\Gamma : \mathfrak{h} \to \mathfrak{h}^*$

be the map given by $h \mapsto (h, \cdot)$. For each $x, y \in \mathfrak{h}^*$, define (x, y) by

$$(x, y) \coloneqq (\Gamma^{-1}(x), \Gamma^{-1}(y)). \tag{2.1.6}$$

Recall that $\{\epsilon_i, \delta_j\}_{i,j}$ is the basis of $\mathfrak{h}^*$ dual to $\{E_{i,i}, E_{m+j,m+j}\}_{i,j}$. We may identify

$$\epsilon_i = \Gamma\left(E_{i,i}\right) \text{ and } \delta_j = -\Gamma\left(E_{m+j,m+j}\right),$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus we have that

$$(\epsilon_i, \epsilon_i) = \delta_{i,j}, \quad (\delta_k, \delta_\ell) = -\delta_{k,\ell} \quad \text{and} \quad (\epsilon_i, \delta_j) = 0, \tag{2.1.7}$$

where $\delta_{i,j}$ stands for the Kronecker delta function.

2.1.3 Weyl Group, Isotropic Roots and Odd Reflections

In this subsection, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$, and Φ be the associated root system of $\mathfrak{g}$. Recall that $\mathfrak{h}$ coincides with the Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. This lets us define the *Weyl group* $\mathcal{W}$ of $\mathfrak{g}$ to be the Weyl group of $\mathfrak{g}_{\bar{0}}$. For example, the Weyl group of $\mathfrak{gl}(m|n)$ is isomorphic to $\mathcal{S}_m \times \mathcal{S}_n$, where $\mathcal{S}_m$ denotes the symmetric group on m letters.

Definition 2.1.5. *Let $\alpha \in \Phi_{\bar{0}}$. The* real reflection *in α, r_α is defined as*

$$r_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \text{ for all } \beta \in \mathfrak{h}^*. \tag{2.1.8}$$

where $(\cdot, \cdot)$ is the bilinear form defined in Equation (2.1.6)

By definition, the real reflections generate the Weyl group. In the case of $\mathfrak{gl}(m|n)$, they permute the roots within $\{\epsilon_i - \epsilon_j\}_{1\le i\ne j\le m}$ and $\{\delta_k - \delta_\ell\}_{1\le k\ne \ell\le n}$ respectively. The odd roots play a vital role in super Lie theory. In particular, we will define *odd reflections* only for certain *simple* roots. Like real reflections, these reflections will take simple systems to simple systems, but unlike real reflections, they are not linear and their action will differentiate the highest weight theory of Lie superalgebras from that of Lie algebras. To formally define odd reflections, we first define isotropic roots.

Definition 2.1.6. *Let Φ be the root system of $\mathfrak{g}$. A root $\alpha \in \Phi$ is called isotropic if $(\alpha, \alpha) = 0$.*

Example 2.1.7. The isotropic roots for $\mathfrak{gl}(m|n)$ are $\pm(\delta_i - \epsilon_j)$ for some $1 \le i \le m$ and $1 \le j \le n$. In fact, they coincide with the set of odd roots in this case. ♠

Definition 2.1.8. *Let Π be a simple system of $\mathfrak{g}$. Let α be a* simple *isotropic root. For any simple root $\beta \in \Pi$, we define*

$$r_\alpha(\beta) = \begin{cases} -\alpha & \text{if } \beta = \alpha, \\ \alpha + \beta & \text{if } (\beta, \alpha) \ne 0, \\ \beta & \text{if } (\beta, \alpha) = 0 \text{ and } \beta \ne \alpha. \end{cases}$$

Then r_α is referred to as the odd reflection with respect to α. Moreover, let $\Pi_\alpha := r_\alpha(\Pi)$.

Remark 2.1.9. In general, not all odd roots are isotropic. For example, δ_1 is a non-isotropic odd root in the root system of $\mathfrak{osp}(2m + 1|2n)$. For the definition of $\mathfrak{osp}(2m + 1|2n)$, we refer the readers to [CW13, Chapter 1].

However, as noted in [CW13, Theorem 1.18.], if $\alpha \in \Phi_{\bar{1}}$ is an odd non-isotropic root, then $2\alpha \in \Phi_{\bar{0}}$ is a (non-isotropic) even root. In this case, following [CW13, Chapter 1.4], one defines r_α to be the reflection in the root 2α.

Note that, an odd reflection is not linear on Φ^+. We illustrate this fact by the following example.

Example 2.1.10. Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and let Π be the standard simple system. Take $\alpha = \beta = \delta_m - \epsilon_1$ and $\gamma = \delta_{m-1} - \delta_m$. If r_α could be extended linearly to Φ^+, we would have $r_\alpha(\beta + \gamma) = r_\alpha(\beta) + r_\alpha(\gamma)$. However, by a straightforward calculation, we have $r_\alpha(\beta) + r_\alpha(\gamma) = \gamma$ and $r_\alpha(\beta + \gamma) = \delta_m + \delta_{m-1} - 2\epsilon_1$ which is not even a root. ♠

2.1.4 Borel subalgebras and $\epsilon\delta$-sequences

One natural question left from the previous subsection is how the odd reflections affect the positive system of $\mathfrak{g}$. To answer this question, we first define the Borel subalgebras of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi = \Phi^+ \cup \Phi^-$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We define

$$\mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha. \tag{2.1.9}$$

Then we have a triangular decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

The solvable subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}^+$ is called the *Borel subalgebra* of $\mathfrak{g}$ corresponding to Φ^+. Notice that for different positive systems (or equivalently different simple systems), one has different Borel subalgebras.

The following result assures us that odd reflections have the same impact on positive systems as real simple reflections. It is given in [CW13, Lemma 1.30], where it is proven case-by-case.

Lemma 2.1.11. *Let Π be a simple system of $\mathfrak{g}$. Let $\alpha \in \Pi$ be an isotropic simple root. Then the set Π_α is a simple system whose positive system, Φ_α^+ is given by*

$$\Phi_\alpha^+ = \{-\alpha\} \cup \Phi^+ \setminus \{\alpha\}. \tag{2.1.10}$$

Unlike the Lie algebra theory, the Borel subalgebras of a Lie superalgebra do not necessarily conjugate by the Weyl group $\mathcal{W}$, since $\mathcal{W}$ is only generated by the *real* reflections. However, all Borel subalgebras are still related in the following sense.

Definition 2.1.12. *Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be two Borel subalgebras of $\mathfrak{g}$. We say $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are related if there exists a sequence of odd and real reflections $r_1, r_2, \dots, r_n$ such that $r_n r_{n-1} \dots r_1(\mathfrak{b}_1) = \mathfrak{b}_2$. The related positive system and simple system are defined in the same way.*

Example 2.1.13. Let $\mathfrak{g} = \mathfrak{gl}(1|2)$. Then $\Pi = \{\delta_1 - \epsilon_1, \epsilon_1 - \epsilon_2\}$ is related to $\Pi' = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \delta_1\}$ by $r_{\epsilon_1-\epsilon_2} r_{\delta_1-\epsilon_2} r_{\epsilon_1-\epsilon_2}$. The intermediate simple systems are

$$\begin{aligned}
\Pi_1 &= r_{\epsilon_1-\epsilon_2}(\Pi) = \{\delta_1 - \epsilon_2, \epsilon_2 - \epsilon_1\}, \\
\Pi_2 &= r_{\delta_1-\epsilon_2}(\Pi_1) = \{\epsilon_2 - \delta_1, \delta_1 - \epsilon_1\}, \\
\Pi' &= r_{\epsilon_1-\epsilon_2}(\Pi_2).
\end{aligned}$$

♠

Proposition 2.1.14. *[CW13, Proposition 1.32] Let $\mathfrak{g}$ be a simple basic Lie superalgebras or $\mathfrak{gl}(m|n)$. Then any two simple systems of $\mathfrak{g}$ are related.*

Now let us fix $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then there is another way to label the disctinct Borel subalgebras containing a given Cartan subalgebra, namely, by $\epsilon\delta$-sequences.

Definition 2.1.15. *Let $\mathfrak{b}$ be an arbitrary Borel subalgebra of $\mathfrak{gl}(m|n)$ containing $\mathfrak{h}$ with associated simple system Π. The $\epsilon\delta$-sequence of $\mathfrak{b}$ or Π is a sequence $\zeta_1 \dots \zeta_{m+n}$*

such that $\zeta_k \in \{\epsilon_i, \delta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ *for each* $k \in \{1, \dots, m+n\}$*, and such that* $\Pi = \{\zeta_k - \zeta_{k+1} \mid 1 \leq k < m+n\}$.

Example 2.1.16. The $\epsilon\delta$-sequence of the standard simple system of $\mathfrak{gl}(m|n)$ is given by $\epsilon_1 \dots \epsilon_m \delta_1 \dots \delta_n$. ♠

Moreover, for all $\sigma \in \mathcal{S}_m$ and $s \in \mathcal{S}_n$, the Borel subalgebras with $\epsilon\delta$-sequences $\epsilon_{\sigma(1)} \dots \epsilon_{\sigma(m)} \delta_{s(1)} \dots \delta_{s(n)}$ are all $\mathcal{W}$-conjugate to the standard Borel subalgebra of $\mathfrak{gl}(m|n)$. Thus we can ignore the indices of an $\epsilon\delta$-sequence to get a $\mathcal{W}$-conjugacy class of Borel algebras of $\mathfrak{gl}(m|n)$. For example,

$$\underbrace{\epsilon \dots \epsilon}_{m} \underbrace{\delta \dots \delta}_{n}$$

is the conjugacy class of the standard Borel subalgebra of $\mathfrak{gl}(m|n)$. In fact, we have the following proposition.

Proposition 2.1.17. *[CW13, Proposition 1.27]Let* $\mathfrak{g} = \mathfrak{gl}(m|n)$*. There exists a one-to-one correspondence between conjugacy classes of Borel subalgebras of* $\mathfrak{g}$ *and the associated* $\epsilon\delta$*-sequences ignoring the index.*

Thus in order to list representatives of all $\mathcal{W}$-conjugacy classes of Borel subalgebras of $\mathfrak{g}$, it suffices to consider only the *increasing Borel subalgebras* in the sense that the corresponding $\epsilon\delta$-sequences have increasing indices among ϵ and δ respectively.

Remark 2.1.18. Notice that by Proposition 2.1.14, any arbitrary increasing Borel subalgebras can be obtained from the standard $\epsilon\delta$-sequence by only applying odd reflections in the following way: Starting from $\epsilon_1 \dots \epsilon_m \delta_1 \dots \delta_n$, first move δ_1 to the left to the desired position through the *simple* odd reflections (that is, roots arising from adjacent elements of the $\epsilon\delta$-sequences) $r_{\epsilon_m - \delta_1}, r_{\epsilon_{m-1} - \delta_1}, \dots,$ and then, move δ_2 to the left to the desired position through the *simple* odd reflections $r_{\epsilon_m-\delta_2}, r_{\epsilon_{m-1}-\delta_2}, \ldots$. Continue this process until for all $1 \leq j \leq n$, δ_j is lying in the desired position.

Example 2.1.19. The $\epsilon\delta$-sequence $\epsilon_1\delta_1\epsilon_2\delta_2\epsilon_3$ is obtained from $\epsilon_1\epsilon_2\epsilon_3\delta_1\delta_2$ by applying $r_{\epsilon_3-\delta_1}$,$r_{\epsilon_2-\delta_1}$ and $r_{\epsilon_3-\delta_2}$ to the standard Borel subalgebra of $\mathfrak{gl}(3|2)$. ♠

2.1.5 The Weyl Vectors

For Lie algebras, the Weyl vector ρ is defined as one half of the sum of the positive roots. We now introduce the Weyl vector for a Lie superalgebra.

Definition 2.1.20. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra with a positive system $\Phi^+ = \Phi^+_{\bar{0}} \cup \Phi^+_{\bar{1}}$. The Weyl vector ρ is defined by*

$$\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$$

where

$$\rho_{\bar{0}} = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\bar{0}}} \alpha \qquad \textit{and} \qquad \rho_{\bar{1}} = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\bar{1}}} \alpha.$$

Example 2.1.21. Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ with Φ^+ being the standard positive system of $\mathfrak{g}$. The set of simple even positive roots is

$$\Phi^+_{\bar{0}} = \{\epsilon_i - \epsilon_j, \delta_k - \delta_\ell\}$$

where $1 \leq i < j \leq m$ and $1 \leq k < \ell \leq n$, and the set of simple odd positive roots is

$$\Phi^+_{\bar{1}} = \{\epsilon_i - \delta_j\}$$

where $1 \leq i \leq m$ and $1 \leq j \leq n$. It follows by direct calculation that the associated

Weyl vector is

$$\rho^{\mathrm{st}} = \sum_{i=1}^{m} \frac{m-n+1-2i}{2}\epsilon_i + \sum_{j=1}^{n} \frac{m+n+1-2j}{2}\delta_j. \qquad \spadesuit$$

Proposition 2.1.22. *Let $\mathfrak{b}$ be an arbitrary Borel subalgebra of a basic Lie superalgebra $\mathfrak{g}$ with associated Weyl vector ρ. Let α be an isotropic simple root with respect to $\mathfrak{b}$. Let $\mathfrak{b}' = r_\alpha(\mathfrak{b})$. Then the Weyl vector ρ' with respect to $\mathfrak{b}'$ is given by $\rho' = \rho + \alpha$.*

Proof. Let $\rho = \rho_{\bar{0}} - \rho_{\bar{1}}$ and $\rho' = \rho'_{\bar{0}} - \rho'_{\bar{1}}$ be the Weyl vectors with respect to $\mathfrak{b}$ and $\mathfrak{b}'$ respectively. Then by Lemma 2.1.11, it is clear that $\rho_{\bar{0}} = \rho'_{\bar{0}}$ and $2\rho'_{\bar{1}} = 2\rho_{\bar{1}} - 2\alpha$. Thus $\rho' = \rho_{\bar{0}} - (\rho_{\bar{1}} - \alpha) = \rho + \alpha$ as claimed. □

We finish this section by giving a general formula for the Weyl vector corresponding to any increasing Borel subalgebra.

Lemma 2.1.23. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Let $\rho_0 = \sum_{i=1}^{m} E_i\delta_i + \sum_{j=1}^{n} F_j\delta_j$ be the standard Weyl vector as in Example 2.1.21. Let $\mathfrak{b}_k$ be an arbitrary increasing Borel subalgebra of $\mathfrak{g}$. Let ρ_k be the associated Weyl vector of $\mathfrak{b}_k$. Let $\epsilon_i - \delta_j$ be any isotropic simple root with respect to $\mathfrak{b}_k$. The ϵ_i and δ_j coefficients of ρ_k are*

$$E_i + j - 1 \qquad \textit{and} \qquad F_j - m + i$$

respectively.

Proof. Since $\epsilon_i - \delta_j$ is a simple isotropic root with respect to $\mathfrak{b}_k$, the $\epsilon\delta$-sequence of $\mathfrak{b}_k$ contains the subsequence

$$\ldots \epsilon_i \delta_j \ldots .$$

In order to get this sequence from the standard $\epsilon\delta$-sequence, we have to move

$\delta_1, \ldots, \delta_{j-1}$ to the left of ϵ_i and $\epsilon_{i+1}, \ldots, \epsilon_m$ to the right of δ_j by a sequence of odd reflections applied to the standard Borel subalgebra. Note that the resulting Weyl vector depends only on the Borel subalgebra and not on the particular sequence of odd reflections. We may therefore take a moving path as described in Remark 2.1.18. Then the odd reflections which involve ϵ_i and δ_j are precisely

$$r_{\epsilon_i-\delta_1}, r_{\epsilon_i-\delta_2}, \ldots, r_{\epsilon_i-\delta_{j-1}} \qquad (j-1\ \epsilon_i's \text{ in total})$$

and

$$r_{\epsilon_m-\delta_j}, r_{\epsilon_{m-1}-\delta_j}, \ldots, r_{\epsilon_{i+1}-\delta_j} \qquad (m-i\ \delta_j's \text{ in total})$$

respectively. Thus by Proposition 2.1.22, the ϵ_i coefficient of ρ_k is $A_i + (j-1)$ and the δ_j coefficient of ρ_k is $B_j - (m-i)$ as claimed. □

2.2 Highest Weight Theory

In this section we give a brief introduction to the representation theory of Lie superalgebras, and specifically including highest weight modules, Young diagrams and the parametrization of irreducible finite dimensional highest weight modules by Young diagrams. In particular, we give two examples of parametrizations in Section 5.1 and 6.1 which play a vital role in the Capelli Eigenvalue Problem.

2.2.1 Lie Superalgbera Modules

In this subsection, we define Lie superalgebra modules, highest weight vectors and the effects of odd reflections on highest weight vectors. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over $\mathbb{C}$. Let $\mathfrak{h}$ a Cartan subalgbera of $\mathfrak{g}$ and Φ a root system of $\mathfrak{g}$ with the positive system Φ^+ and $\mathfrak{b} = \mathfrak{g}^+ \oplus \mathfrak{h}$ a Borel subalgebra of $\mathfrak{g}$, where $\mathfrak{g}^+$ is defined in Equation (2.1.9).

Definition 2.2.1. *A* $\mathfrak{g}$-module *is a complex super vector space* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *together with a* $\mathfrak{g}$*-action*

$$\mathfrak{g} \times V \to V, \ (x, v) \mapsto xv$$

which is a bilinear map satisfying the following properties:

(i) If $x \in \mathfrak{g}_i$ *and* $v \in V_j$*, then* $xv \in V_{i+j}$ *for all* $i, j \in \mathbb{Z}_2$*,*

(ii) $[x, y]v = x(yv) - (-1)^{|i|+|j|} y(xv)$*, for all* $x \in \mathfrak{g}_i, \ y \in \mathfrak{g}_j$ *and for all* $v \in V$*.*

Recall that a $\mathfrak{g}$-module V is called *simple* if it has no nontrivial $\mathbb{Z}_2$-graded submodules.

Definition 2.2.2. *Let* V *be a* $\mathfrak{g}$*-module. A vector* v *in* V *is a* $\mathfrak{b}$-highest weight vector *of weight* $\underline{\lambda} \in \mathfrak{h}^*$ *if the following holds:*

(i) $Xw = 0$*, for all* $X \in \mathfrak{g}^+$*,*

(ii) $Hw = \underline{\lambda}(H)w$ *for all* $H \in \mathfrak{h}$*.*

Next we give two lemmas which will be used in computing the weight of highest weight vectors with respect to different Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$.

Lemma 2.2.3. *Let* α *be a simple isotropic root and* v *be a* $\mathfrak{b}$*-highest weight vector for a module* V *with respect to positive system* Φ^+*. Let* $e_\alpha \in \mathfrak{g}_\alpha$ *and* $f_\alpha \in \mathfrak{g}_{-\alpha}$*. Then*

(i) $e_\alpha f_\alpha(v) = [e_\alpha, f_\alpha](v)$ *and* $e_\beta f_\alpha(v) = [e_\beta, f_\alpha](v) = 0$ *for all* $\beta \in \Phi^+ \setminus \{\alpha\}$*;*

(ii) $f_\alpha^2(v) = 0$*.*

Proof. We have

(i) $[e_\alpha, f_\alpha](v) = e_\alpha f_\alpha(v) - (-1)^{|e_\alpha||f_\alpha|} f_\alpha e_\alpha(v) = e_\alpha f_\alpha(v)$ since v is highest weight vector so that $f_\alpha e_\alpha(v) = f_\alpha(0) = 0$. Similarly, we deduce that $e_\beta f_\alpha(v) = [e_\beta, f_\alpha](v)$. Also by definition we have $[e_\beta, f_\alpha] \subseteq \mathfrak{g}_{\beta-\alpha}$. Since α is a simple root and β is a positive root, $\beta - \alpha$ is either not a root or it is in $\Phi^+ \setminus \{\alpha\}$. Thus $[e_\beta, f_\alpha](v) = 0$.

(ii) Notice that since α is odd, $f_\alpha \in \mathfrak{g}_{-\alpha}$ is odd. We have that

$$[f_\alpha, f_\alpha](v) = \left(f_\alpha\left(f_\alpha v\right) - (-1)^{|f_\alpha||f_\alpha|} f_\alpha\left(f_\alpha v\right)\right) = 2 f_\alpha\left(f_\alpha v\right)$$

which implies that $f_\alpha\left(f_\alpha v\right) = \frac{1}{2}[f_\alpha, f_\alpha](v)$. However, since $[f_\alpha, f_\alpha] \in \mathfrak{g}_{-2\alpha} = \{0\}$. Thus, $f_\alpha^2(v) = 0$ as claimed. □

Lemma 2.2.4. *Let V be a simple $\mathfrak{g}$-module. Let $v \in V$ be a $\mathfrak{b}$-highest weight vector of highest weight $\underline{\lambda}$ with respect to some positive system Φ^+. Let α be a simple isotropic root and r_α be the odd reflection with respect to α. Let $h_\alpha := [e_\alpha, f_\alpha]$ where $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ are nonzero weight vectors. Let $\mathfrak{b}_\alpha = r_\alpha(\mathfrak{b})$.*

(i) If $\underline{\lambda}(h_\alpha) = 0$, then v is a $\mathfrak{b}_\alpha$-highest weight vector with $\mathfrak{b}_\alpha$-highest weight $\underline{\lambda}$.

(ii) If $\underline{\lambda}(h_\alpha) \neq 0$, then $f_\alpha v$ is a $\mathfrak{b}_\alpha$-highest weight vector with $\mathfrak{b}_\alpha$-highest weight $\underline{\lambda} - \alpha$.

Proof. First notice that $[e_\alpha, f_\alpha]v = h_\alpha v = \underline{\lambda}(h_\alpha)v$. To show a vector w is a $\mathfrak{b}_\alpha$-highest weight vector is equivalent to showing $e_\beta w = 0$ for all $\beta \in \Phi^+_\alpha$.

(i) We first assume $\underline{\lambda}(h_\alpha) = 0$. Since by assumption v is a highest weight vector with respect to Φ^+, we have $e_\beta v = 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$. It suffices to show $f_\alpha v = 0$. Suppose $f_\alpha v \neq 0$. But by Lemma 2.2.3 (i) we have $e_\alpha\left(f_\alpha\right) v =$

$[e_\alpha, f_\alpha]v = \underline{\lambda}(h_\alpha)v = 0$. Also by Lemma 2.2.3 (ii), we have $e_\beta\,(f_\alpha)\,v = 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$. Thus $f_\alpha v$ is another $\mathfrak{b}$-highest weight vector with weight $\underline{\lambda} - \alpha < \underline{\lambda}$. This contradicts the uniqueness of highest weight. Therefore $f_\alpha v = 0$ and hence v is a $\mathfrak{b}_\alpha$-highest weight vector with $\mathfrak{b}_\alpha$-highest weight $\underline{\lambda}$.

(ii) Now assume $\underline{\lambda}(h_\alpha) \neq 0$. Then $e_\alpha\,(f_\alpha v) \neq 0$ implies $f_\alpha v \neq 0$. By Lemma 2.2.3 (ii), we have $e_\beta(f_\alpha v) = 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$. Also by Lemma 2.2.3 (iii), we have $f_\alpha(f_\alpha v) = 0$ which implies that $f_\alpha v$ is a $\mathfrak{b}_\alpha$-highest weight vector. Moreover, as noted above, f_α has weight $-\alpha$, it is a simple calculation to see the weight of $f_\alpha v$ is $\underline{\lambda} - \alpha$. □

In fact, we can compute the conditions of this lemma directly. With respect to our bilinear form on $\mathfrak{h} \times \mathfrak{h}$, there is an element h_α in $\mathfrak{h}$ such that for all $\underline{\lambda} \in \mathfrak{h}^*$, $(\underline{\lambda}, \alpha) = \underline{\lambda}(h_\alpha)$, and up to scaling this coincides with $[e_\alpha, f_\alpha]$. Therefore we have the following useful corollary.

Corollary 2.2.5. *Retain the set-up in Lemma 2.2.4. We have*

(i) If $(\underline{\lambda}, \alpha) = 0$, then v is a $\mathfrak{b}_\alpha$-highest weight vector with $\mathfrak{b}_\alpha$-highest weight $\underline{\lambda}$.

(ii) If $(\underline{\lambda}, \alpha) \neq 0$, then $f_\alpha v$ is a $\mathfrak{b}_\alpha$-highest weight vector with $\mathfrak{b}_\alpha$-highest weight $\underline{\lambda} - \alpha$.

2.2.2 Partitions and Young Diagrams

In this subsection, we define partitions and Young diagrams. We give an order on the set of partitions which will be useful in later chapters. Then we define an important class of partitions called hook partitions.

Definition 2.2.6. *By a partition of $k \in \mathbb{Z}_{\geq 0}$, we mean a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ satisfying $|\lambda| = \sum_{i \geq 0} \lambda_i = k$. We call $|\lambda|$ the* *size of* λ*. Moreover, we denote the length of* λ*,* $\ell(\lambda)$ *to be the index of the last nonzero entry of* λ*. We denote the set of all partitions such that* $\ell(\lambda) \leq n$ *by* $\mathcal{P}_n$*.*

By convention, we assume two partitions which differ only by a string of zeros at the end are the same. For example, we regard $(2,1), (2,1,0), (2,1,0,0,\ldots)$ as the same partition.

Definition 2.2.7. *For any two partitions* μ *and* λ*, we say* $\lambda \succeq \mu$ *if* $|\lambda| > |\mu|$*, or* $|\lambda| = |\mu|$ *and the first non-vanishing difference* $\lambda_i - \mu_i$ *is positive.*

For example, we have

$$\varnothing \prec (1) \prec (1,1) \prec (2) \prec (1,1,1) \prec (2,1) \prec (3).$$

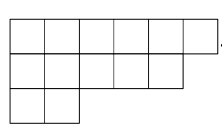
There is also a partial order which will be useful later.

Definition 2.2.8. *For any two partitions* μ *and* λ*, we say* $\lambda \geq \mu$ *if* $|\lambda| > |\mu|$*, or* $|\lambda| = |\mu|$ *and* $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ *for all* $i \geq 1$*.*

Notice that it is straightforward that if $\lambda \geq \mu$, then $\lambda \succeq \mu$. That is, the total order $\succeq$ is compatible with $\geq$.

Definition 2.2.9. *Given a partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$*, the Young diagram of shape* λ *is a top-left-aligned diagram with* $\ell(\lambda)$ *rows of boxes, and* λ_i *boxes in the i-th row.*

Example 2.2.10. The Young diagram for $\lambda = (6,5,2,0,0)$ is

. ♠

It is straightward from the definition that there is a one-to-one correspondence between the set of partitions and the set of Young diagrams. Therefore we abuse the notation to let λ indicate both partitions and Young diagram.

The set of partitions (thus Young diagrams) plays a vital role in the decomposition of Lie algebra modules. In the super-setting, the set of hook partitions plays an analogous role.

Definition 2.2.11. *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. We say λ is an (m, n)-hook partition if $\lambda_{m+1} \leq n$. We denote the set of all hook partitions by $\mathcal{H}(m, n)$.*

Note that any Young diagram which can be fitted in Figure 2.1 is called an (m, n)-hook diagram. Thus the Young diagram in Example 2.2.10 is a $(2, 3)$-hook tableau.

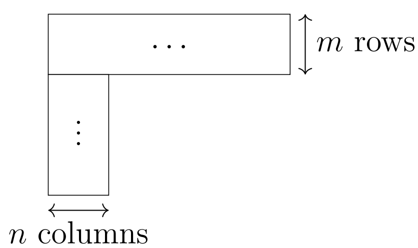


Figure 2.1: (m, n)-hook shape Young diagram frame

By [CW01, Proposition 2.2], the complete list of pairwise non-isomorphic finite-dimensional irreducible $\mathfrak{gl}(m|n)$-modules is indexed by their highest weights

$$\underline{\lambda} = \sum_{i=1}^{m} \lambda_i \epsilon_i + \sum_{j=1}^{n} \mu_j \delta_j$$

such that $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ and $\mu_j - \mu_{j+1} \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Furthermore, the set of (m, n)-hook partitions parametrizes the class of finite-dimensional irreducible $\mathfrak{gl}(m|n)$-modules of polynomial type by [CW01, Proposition

3.26]. We finish this chapter by giving this parametrization. We first give a formal definition of the transpose of a partition.

Definition 2.2.12. *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition size k and length $\ell(\lambda)$. The transpose of λ, denoted $\lambda' = (\lambda_1', \lambda_2', \ldots)$ is a partition of size k and length $\ell(\lambda') = \lambda_1$, where $\lambda_j' \coloneqq \mathrm{Card}\{i \mid \lambda_i - j \geq 0\}$, for all $j \geq 1$.*

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Let $\mathfrak{b}$ be the standard upper triangular Borel subalgebra of $\mathfrak{g}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{H}(m, n)$. Let $V_{m|n}^{\lambda}$ denote the irreducible $\mathfrak{g}$-module with $\mathfrak{b}$-highest weight $\underline{\lambda}$. Then the highest weight $\underline{\lambda}$ parametrized by $\lambda \in \mathcal{H}(m, n)$ is given by

$$\underline{\lambda} = \sum_{i=1}^{m} \lambda_i \epsilon_i + \sum_{j=1}^{n} \langle \lambda_j' - m \rangle \delta_j$$

where $\langle x \rangle \coloneqq \max\{x, 0\}$ for all $x \in \mathbb{R}$. For more details, we refer the readers to [CW01].

Example 2.2.13. Let $\mathfrak{g} = \mathfrak{gl}(2|5)$. Let $\lambda = (6, 5, 2, 0, 0) \in \mathcal{H}(2, 3)$. Then we have that $(\lambda_1', \lambda_2', \lambda_3', \lambda_4', \lambda_5', \lambda_6') = (3, 3, 2, 2, 2, 1)$. Thus the highest weight $\underline{\lambda}$ of $V_{m|n}^{\lambda}$ is given by

$$6\epsilon_1 + 5\epsilon_2 + \delta_1 + \delta_2.$$

In this thesis, we often write such a highest weight as $\underline{\lambda} = (6, 5|1, 1, 0, 0, 0)$. ♠

We shall see that the set $\mathcal{H}(m, n)$ parametrizes the decomposition of a very particular $\mathfrak{gl}(m|n)$-module into irreducible submodules in Chapter 5, and the subset of $\mathcal{H}(m, 2n)$, which consists all $(m, 2n)$-hook partitions such that each part is even is the parameter set in Chapter 6.

We have now finished our preliminary chapters. From the next chapter and onwards, we focus on the Capelli Eigenvalue Problem and finding the refined solution to the CEP with respect to different Borel subalgebras.

Chapter 3

An introduction to the Capelli Eigenvalue Problem

We begin this chapter with introducing the category of super vector spaces, the super analogues of the symmetric and polynomial algebras in this category, and the constant and polynomial coefficient differential operators. Then we carefully define the Capelli Eigenvalue Problem.

3.1 Tensor Product of Super Vector Spaces

In this section, we briefly discuss the category of super vector spaces. We also extend some general properties of tensor products to the super setting. Let V and W be two super vector spaces. Recall that a linear transformation $f : V \to W$ is called *grading preserving* if $f(V_i) \subseteq W_i$ for all $i \in \mathbb{Z}_2$, that is f has parity $\bar{0}$. Denote `SVect` the category whose objects are super vector spaces, that is, vector space graded by $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, and whose morphisms are the grading preserving linear transformations.

Then `SVect` is a symmetric monoidal category. The category `SVect` is monoidal since the tensor product $V \otimes W$ is an object in `SVect` with grading

$$(V \otimes W)_i = \bigoplus_{j+k=i} (V_j \otimes W_k) \text{ for all } i, j, k \in \mathbb{Z}_2. \tag{3.1.1}$$

The category `SVect` is symmetric since $V \otimes W \cong W \otimes V$ as vector super spaces by the isomorphism

$$\begin{aligned} f_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto (-1)^{|v||w|} w \otimes v, \end{aligned}$$

where we denote by $|v|$ the parity of the homogeneous element $v \in V$. Throughout this paper, we may use $|v|$ directly, which implicitly assumes that v is homogeneous. Also notice that the map $f_{V,W}$ clearly preserves the grading, and hence it is a morphism in `SVect`. Moreover, $\text{Hom}(V, W)$, the space of all linear transformations from V to W is also an object of `SVect` with grading

$$\text{Hom}(V, W)_i = \{f \in \text{Hom}(V, W) \mid f(V_j) \subseteq W_{i+j}\} \text{ for all } i, j \in \mathbb{Z}_2.$$

Thus the dual space V^* of a super vector space V defined as

$$V^* := \text{Hom}(V, \mathbb{C}^{1|0})$$

is also a super vector space.

We now show that $W \otimes V^* \to \text{Hom}(V, W)$ is an isomorphism in `SVect`. Let

$T_{w\otimes v^*}: V \to W$ be the map defined by

$$T_{w\otimes v^*}(v) = wv^*(v) \text{ for all } v \in V \tag{3.1.2}$$

extended by linearity. Then we have the following lemma.

Lemma 3.1.1. *The map*

$$W \otimes V^* \to \mathrm{Hom}(V, W) \textit{ such that } w \otimes v^* \mapsto T_{w\otimes v^*} \tag{3.1.3}$$

is an isomorphism in `SVect`.

Proof. The space $V^* \otimes W$ and $\mathrm{Hom}(V, W)$ have the same dimension, and the map is surjective: let $F \in \mathrm{Hom}(V, W)$. Choose a basis $\{v_1, \ldots, v_k\}$ for V and let $\{v_1^*, \ldots, v_k^*\}$ be the dual basis of V^*. Then for $v \in V$, we have $v = \sum_{i=1}^k c_i v_i$ where $c_i = v_i^*(v)$ for each $i = 1, \ldots, n$. Thus we have

$$F(v) = F\left(\sum_{i=1}^k c_i v_i\right) = \sum_{i=1}^k F(v_i)c_i = \sum_{i=1}^k F(v_i)v_i^*(v) = \left(\sum_{i=1}^k T_{F(v_i)\otimes v_i^*}\right)(v)\,,$$

and hence F is in the image of the map defined in Equation 3.1.3. Moreover, we observe that the map in Equation (3.1.3) is grading preserving by noticing v_i and v_i^* have the same parity if they are homogenous. □

3.2 The Super Symmetric Algebra and The Superpolynomial Algebra

Let S_d be the symmetric group on d letters. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space. Then the natural representation of S_d on $V^{\otimes d}$ is defined by the following: let $\sigma \in S_d$. Then the representation is defined by

$$\sigma \mapsto T^{\sigma}_{V,d}$$

for $T^{\sigma}_{V,d}$ defined by

$$T^{\sigma}_{V,d}\left(v_1 \otimes \cdots \otimes v_d\right) = (-1)^{\epsilon\left(\sigma^{-1}: v_1 \otimes \cdots \otimes v_d\right)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)} \tag{3.2.1}$$

for any homogenous simple tensor $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$, where

$$\epsilon\left(\sigma : v_1 \otimes \cdots \otimes v_d\right) = \sum_{\substack{1 \leq r < s \leq d \\ \sigma(r) > \sigma(s)}} \left|v_{\sigma(r)}\right|\left|v_{\sigma(s)}\right|$$

for all $\sigma \in S_d$, homogenous $v_i \in V$ and $1 \leq i \leq d$. As usual we can extend this map uniquely by linearity to a representation of S_d on $V^{\otimes d}$.

Example 3.2.1. Consider $\sigma = (1, 2) \in S_2$ and $V = \mathbb{C}^{1|1}$ with basis $\{e_{\bar{0}}, e_{\bar{1}}\}$. Then

$$T^{\sigma}_{V,2}\left(e_{\bar{0}} \otimes e_{\bar{1}}\right) = (-1)^{|e_{\bar{0}}||e_{\bar{1}}|} e_{\bar{1}} \otimes e_{\bar{0}} = e_{\bar{1}} \otimes e_{\bar{0}}.$$

♠

Now define $\mathtt{Sym}_V^d : V^{\otimes d} \to V^{\otimes d}$ by

$$\mathtt{Sym}_V^d = \frac{1}{|S_d|} \sum_{\sigma \in S_d} T_{V,d}^{\sigma}.$$

That is $\mathtt{Sym}_V^d \left(V^{\otimes d}\right)$ is the space of supersymmetric tensors in $V^{\otimes d}$.

Definition 3.2.2.

1. *The super symmetric algebra $\mathcal{S}(V)$ of V is defined as*

$$\mathcal{S}(V) := \bigoplus_{d \geq 0} \mathcal{S}^d(V) \text{ where } \mathcal{S}^d(V) := \mathtt{Sym}_V^d \left(V^{\otimes d}\right).$$

2. *The super polynomial algebra $\mathcal{P}(V)$ on V is defined as*

$$\mathcal{P}(V) := \mathcal{S}(V^*) = \bigoplus_{d \geq 0} \mathcal{P}^d(V) \text{ where } \mathcal{P}^d(V) := \mathcal{S}^d(V^*).$$

From the definition, we clearly have $\mathcal{S}^1(V) = V$ and $\mathcal{P}^1(V) = V^*$. Now consider η in $V_{\bar{0}}^*$. Then η extends to a linear functional on $V^{\otimes d}$ by $(\eta \otimes \cdots \otimes \eta)(v_1 \otimes \cdots \otimes v_d) = \eta(v_1) \cdots \eta(v_n)$, where linearity follows from the linearity of η. This implies that every $\eta \in V_{\bar{0}}^*$ extends to a homomorphism $\mathtt{h}_\eta : \mathcal{S}(V) \to \mathbb{C}^{1|0}$ such that

$$\mathtt{h}_\eta \left(\mathtt{Sym}_V^d (v_1 \otimes \cdots \otimes v_d)\right) = \frac{1}{|S_d|} \sum_{\sigma \in S_d} \eta(v_1) \cdots \eta(v_d) = \eta(v_1) \cdots \eta(v_d).$$

Now let $\mathfrak{g}$ be a Lie superalgebra. Let (ρ, V) be a $\mathfrak{g}$-module. Then the $\mathfrak{g}$-invariant subspace of V is defined by

$$V^{\mathfrak{g}} := \{v \in V \mid \rho(x)v = 0\}$$

for all x in $\mathfrak{g}$. Let (π, W) be another $\mathfrak{g}$-module. Then, $\mathrm{Hom}(V, W)$, the set of all linear transformations from V to W is also a $\mathfrak{g}$-module with the $\mathfrak{g}$-module structure given by

$$x \cdot T(v) = \pi(x)\,(T(v)) - (-1)^{|T||x|} T(\rho(x)v) \tag{3.2.2}$$

for all $x \in \mathfrak{g}, T \in \mathrm{Hom}\,(V, W)$ and $v \in V$. The set of super intertwiners defined as

$$\mathrm{Hom}_{\mathfrak{g}}\,(V, W) \coloneqq \{T \in \mathrm{Hom}(V, W) \mid \pi(x)T(v) = (-1)^{|T||x|} T\,(\rho(x)v)\}$$

is the set of homomorphisms from V to W that respect their $\mathfrak{g}$-modules structures.

Lemma 3.2.3. *As $\mathfrak{g}$-modules, we have* $\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}(V, W)^{\mathfrak{g}}$.

Proof. Let (ρ, V) and (π, W) be two $\mathfrak{g}$-modules. For all x in $\mathfrak{g}$ and v in V, we have that

$$\begin{aligned} T \in \mathrm{Hom}(V, W)^{\mathfrak{g}} &\text{ if and only if } x \cdot T(v) = 0 \\ &\text{ if and only if } \pi(x)\,(T(v)) - (-1)^{|T||x|} T(\rho(x)v) = 0 \\ &\text{ if and only if } \pi(x)\,(T(v)) = (-1)^{|T||x|} T(\rho(x)v) \\ &\text{ if and only if } T \in \mathrm{Hom}_{\mathfrak{g}}(V, W). \end{aligned}$$

Thus we have $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$, and in particular $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{End}(V)^{\mathfrak{g}}$. □

Similarly as in Subsection 3.1, the category of $\mathfrak{g}$-modules is also a symmetric monoidal category. For two objects in this category, the set of morphisms from V to W can be identified with $\mathrm{Hom}_{\mathfrak{g}}(V, W)_{\bar{0}}$.

3.3 Constant and Polynomial Coefficient Differential Operators

In this section, we define the superderivations of V, the algebra of constant and polynomial coefficient differential operators of V, denoted as $\mathcal{D}(V)$ and $\mathcal{PD}(V)$ repsectively. Next we construct an isomorphism $\mathcal{PD}(V) \cong \mathcal{P}(V) \otimes \mathcal{S}(V)$ as super vector spaces. We first recall the definition of the contragredient module.

Definition 3.3.1. *Let W be a $\mathfrak{g}$-module. The contragredient module of W is the super vector space $W^* := \{f : W \to \mathbb{C} \mid f$ is a linear functional $\}$, such that*

1. *the $\mathbb{Z}_2$-grading of W is defined by $(W^*)_i := \{f \in W^* \mid f(W_j) = 0$ if $i \neq j\}$,*
2. *the $\mathfrak{g}$-action is defined on homogeneous elements by*

$$(xf)(w) = -(-1)^{|x||f|} f(xw).$$

Definition 3.3.2. *Let W be a super vector space. Let $\mathcal{P}(W)$ be the algebra of superpolynomials of W. For every homogenous $w \in W$, the superderivation ∂_w of $\mathcal{P}(W)$ with parity $|\partial_w| = |w|$ is defined uniquely by*

$$\partial_w (v^*) := (-1)^{|w||v^*|} \langle v^*, w \rangle \text{ for all homogenous } v^* \in W^* \cong \mathcal{P}^1(W). \tag{3.3.1}$$

We then extend this definition to all polynomials in $\mathcal{P}(W)$ by

$$\partial_w(ab) = \partial_w(a)b + (-1)^{|w||a|} a\partial_w(b)$$

for all homogenous elements $w \in W$ and $a \in \mathcal{P}(W)$.

Remark 3.3.3. By Definition 3.3.1, $\partial_w(v^*)$ is nonzero only if $|w| = |v^*|$. Thus Equation (3.3.1) can be rewritten as $\partial_w(v^*) \coloneqq (-1)^{|w|}\langle v^*, w\rangle$

For homogenous $w \in W = \mathcal{S}^1(W)$, we defined ∂_w. By the universality of the tensor algebra $\mathcal{T}(W)$, we can extend the map $\partial : W \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}(W))$ to a homomorphism of associative algebras $\partial : \mathcal{T}(W) \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}(W))$. Therefore we have the following lemma.

Lemma 3.3.4. *The map $\partial : \mathcal{T}(W) \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}(W))$ descends to a homomorphism of associative algebras $\partial : \mathcal{S}(W) \to \mathrm{End}_{\mathbb{C}}(\mathcal{P}(W))$*

Proof. It suffices to check that the derivations are supercommutive. That is, we show that $\partial_{w_1}\partial_{w_2} = (-1)^{|w_1||w_2|}\partial_{w_2}\partial_{w_1}$ for all elements in $\mathcal{P}(V)$, and hence the above definition is well-defined. Let $w_1, w_2 \in W$ and $a, b \in \mathcal{P}(W)$ be homogenous. Then we have

$$\begin{aligned}\partial_{w_1}\partial_{w_2}(ab) &= \partial_{w_1}\left(\partial_{w_2}(a)b + (-1)^{|w_2||a|}a\partial_{w_2}(b)\right)\\ &= \partial_{w_1}\partial_{w_2}(a)b + (-1)^{|w_1|(|w_2|+|a|)}\partial_{w_2}(a)\partial_{w_1}(b)\\ &\quad + (-1)^{|w_2||a|}\left(\partial_{w_1}(a)\partial_{w_2}(b) + (-1)^{|w_1||a|}a\partial_{w_1}\partial_{w_2}(b)\right).\end{aligned}$$

Similarly, we have that $(-1)^{|w_1||w_2|}\partial_{w_2}\partial_{w_1}(ab)$ equals to

$$\begin{aligned}&(-1)^{|w_1||w_2|}\left(\partial_{w_2}\left(\partial_{w_1}(a)b + (-1)^{|w_1||a|}a\partial_{w_1}(b)\right)\right)\\ &= (-1)^{|w_1||w_2|}\left(\partial_{w_2}\partial_{w_1}(a)b + (-1)^{|w_2|(|w_1|+|a|)}\partial_{w_1}(a)\partial_{w_2}(b)\right)\\ &\quad + (-1)^{|w_1||w_2|}\left((-1)^{|w_1||a|}\left(\partial_{w_2}(a)\partial_{w_1}(b) + (-1)^{|w_2||a|}a\partial_{w_2}\partial_{w_1}(b)\right)\right)\\ &= \partial_{w_1}\partial_{w_2}(a)b + (-1)^{|w_1||w_2|}(-1)^{|w_2|(|w_1|+|a|)}\partial_{w_1}(a)\partial_{w_2}(b)\\ &\quad + (-1)^{|w_1||a|}\left((-1)^{|w_1||w_2|}\partial_{w_2}(a)\partial_{w_1}(b) + (-1)^{|w_2||a|}a\partial_{w_1}\partial_{w_2}(b)\right)\end{aligned}$$

$$= \partial_{w_1}\partial_{w_2}(a)b + (-1)^{|w_2||a|}\partial_{w_1}(a)\partial_{w_2}(b)$$
$$+ (-1)^{|w_1|(|w_2|+|a|)}\partial_{w_2}(a)\partial_{w_1}(b) + (-1)^{|w_2||a|}(-1)^{|w_1||a|}a\partial_{w_1}\partial_{w_2}(b).$$

Thus, we have that

$$\partial_{w_1}\partial_{w_2}(ab) = (-1)^{|w_1||w_2|}\partial_{w_2}\partial_{w_1}(ab). \tag{3.3.2}$$

Thus the result follows from an induction on the degree of $f \in \mathcal{P}(V)$ and the fact every polynomial $f \in \mathcal{P}(V)$ is a linear combination of monomials. □

Lemma 3.3.5. *Let $\mathcal{D}(W)$ be the algebra of all constant-coefficient differential operators on W. Then the map*

$$\mathcal{S}(W) \to \mathcal{D}(W) \text{ that sends } s \mapsto \partial_s$$

is an algebera isomorphism.

Proof. The result follows from Lemma 3.3.4. □

Recall that $\mathcal{S}(W)$ is a $\mathfrak{g}$-module whose action is defined canonically from the action of $\mathfrak{g}$ on W and extended by the Leibniz rule defined in Definition 2.2.1 (ii). Since $\mathcal{D}(W) \cong \mathcal{S}(W)$ as vector spaces according to Lemma 3.3.5, $\mathcal{D}(W)$ inherits this action. We make this precise in the following lemma.

Lemma 3.3.6. *Let (π, W) be a $\mathfrak{g}$-module. Let $\mathcal{D}(W)$ be the algebra of all constant-coefficient differential operators on W. Then $(\pi_{\mathcal{D}}, \mathcal{D}(W))$ is a $\mathfrak{g}$-module with action defined by $\pi_{\mathcal{D}}(x)\partial_w := \partial_{\pi(x)w}$ for all $x \in \mathfrak{g}$ and $w \in W$. The action extends from W to $\mathcal{S}(W)$ canonically.*

Proof. Because $\mathcal{D}(W)$ inherits the action of $\mathfrak{g}$ on $\mathcal{S}(W)$, it suffices to prove the lemma for homogenous $x, y \in \mathfrak{g}$ and $w \in W$. First since (π, W) is a $\mathfrak{g}$-module, we have $\pi(x)w$ has parity $|x| + |w|$. Thus $\pi_{\mathcal{D}}(x)\partial_w = \partial_{\pi(x)w}$ has parity $|x| + |w|$ as well. Let $v^* \in W^*$ be homogeneous; we then have

$$\begin{aligned}
&\pi_{\mathcal{D}}(x)\pi_{\mathcal{D}}(y)\partial_w(w^*) - (-1)^{|x||y|}\pi_{\mathcal{D}}(y)\pi_{\mathcal{D}}(x)\partial_w(w^*) \\
&= \partial_{\pi(x)\pi(y)w}(v^*) - (-1)^{|x||y|}\partial_{\pi(y)\pi(x)w}(v^*) \\
&= (-1)^{|v^*|(|x|+|y|+|w|)}\langle v^*, \pi(x)\pi(y)w\rangle - (-1)^{|x||y|(|x|+|y|+|w|)}\langle v^*, \pi(y)\pi(x)w\rangle \\
&= (-1)^{|v^*|(|x|+|y|+|w|)}\langle v^*, \pi(x)\pi(y)w - (-1)^{|x||y|}\pi(y)\pi(x)w\rangle \\
&= (-1)^{|v^*|(|x|+|y|+|w|)}\langle v^*, \pi([x, y])w\rangle \\
&= \partial_{\pi([x,y])w}(v^*) \qquad \text{(by the fact that } [x, y] \text{ has parity } |x| + |y|) \\
&= \pi_{\mathcal{D}}\left([x, y]\right)\partial_w(v^*).
\end{aligned}$$

which completes the proof. □

Recall that in Lemma 3.3.5, we showed that $\mathcal{S}(W) \cong \mathcal{D}(W)$ is an algebra isomorphism. We then show that $\mathcal{S}(W) \cong \mathcal{D}(W)$ as $\mathfrak{g}$-modules.

Proposition 3.3.7. *Let (π, W) be a $\mathfrak{g}$-module. Let $\mathcal{D}(W)$ be the algebra of all constant-coefficient differential operators on W. Then*

$$\mathcal{S}(W) \to \mathcal{D}(W) \text{ such that } s \mapsto \partial_s$$

gives a $\mathfrak{g}$-module isomorphism.

Proof. Since $s \in W$ and $\partial_s \in \mathcal{D}(W)$ generate $\mathcal{S}(W)$ and $\mathcal{D}(W)$ respectively, it suffices to consider the action of homogenous $x \in \mathfrak{g}$ on homogenous elements $s \in W$. Let f

be the map defined in Lemma 3.3.5. Then we have

$$f(\pi(x)s) = \partial_{\pi(x)s} = \pi_{\mathcal{D}}(x)\partial_s = \pi_{\mathcal{D}}(x)f(s). \qquad \square$$

Definition 3.3.8. *The associative super algebra of polynomial-coefficient differential operators on W, denoted by $\mathcal{PD}(W)$, is the subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathcal{P}(W))$ given by*

$$\mathcal{PD}(W) = \operatorname{span}_{\mathbb{C}}\{a\partial_b \mid a \in \mathcal{P}(W),\ b \in \mathcal{S}(W)\}.$$

It is straightforward that if W is a $\mathfrak{g}$-module, then $\mathcal{PD}(W) \subset \operatorname{End}_{\mathbb{C}}(\mathcal{P}(W))$ is $\mathfrak{g}$-invariant subspace under the action of $\mathfrak{g}$ given by Equation (3.2.2). We next construct an isomorphism $\mathcal{PD}(V) \cong \mathcal{P}(V) \otimes \mathcal{S}(V)$. We first give an action of $\mathfrak{g}$ on $\mathcal{D}(W)$.

Notice that we can consider ∂_w as an element in $\operatorname{End}_{\mathbb{C}}(\mathcal{P}(W))$. Thus $\mathcal{D}(W)$ is a $\mathfrak{g}$-module whose action is given by Equation (3.2.2). Our next goal is to show that this action and the action defined in Lemma 3.3.6 agree with each other.

Lemma 3.3.9. *Let $x \in \mathfrak{g}$, $w \in W$ and $v^* \in W^*$ be homogenous. Then with respect to the action define in Equation* (3.2.2), *$(\pi_{\tilde{\mathcal{D}}}, \mathcal{D}(W))$ is a $\mathfrak{g}$-module with*

$$(\pi_{\tilde{\mathcal{D}}}(x)\partial_w)(v^*) = (-1)^{|v^*|(|w|+|x|)}\langle v^*, \pi(x)w\rangle.$$

Proof. Let $x \in \mathfrak{g}$, $w \in W$ and $v^* \in W^*$ be homogenous. Since $\pi_{\tilde{D}}(x) \in \operatorname{End}(\mathcal{P}(W))$, the action π and ρ defined in Equation (3.2.2) are π^*, the contragredient module of W. Thus we have

$$\begin{aligned}(\pi_{\tilde{\mathcal{D}}}(x)\partial_w)(v^*) &= \pi^*(x)\left(\partial_w(v^*)\right) - (-1)^{|x||w|}\partial_w\left(\pi^*(x)v^*\right) && \text{by (3.2.2)}\\ &= \pi^*(x)\left((-1)^{|w|}\langle v^*, w\rangle\right) - (-1)^{|x||w|}(-1)^{|w|(|x|+|v^*|)}\langle \pi^*(x)v^*, w\rangle && \text{by (3.3.1)}\end{aligned}$$

where the first term of the right hand side is 0 by the fact $\langle v^*, w\rangle = v^*(w)$ is constant. To simplify the second term, recall that the contragredient module (π^*, W^*) action implies that

$$\langle \pi^*(x)v^*, w\rangle = -(-1)^{|x||v^*|}\langle v^*, \pi(x)w\rangle. \tag{3.3.3}$$

Thus by Equation (3.3.3) , the second term simplifies to

$$\begin{aligned} -(-1)^{|x||w|}(-1)^{|w|(|x|+|v^*|)}(-1)(-1)^{|x||v^*|}\langle \pi^*(x)v^*, w\rangle &= (-1)^{|v^*||w|+|x||v^*|}\langle v^*, \pi(x)w\rangle \\ &= (-1)^{|v^*|(|w|+|x|)}\langle v^*, \pi(x)w\rangle. \end{aligned}$$

which completes the proof. □

Proposition 3.3.10. *Let (π, W) be a $\mathfrak{g}$-module. The $\mathfrak{g}$ actions on $\mathcal{D}(W)$ defined by its isomorphism with $\mathcal{S}(W)$ (Lemma 3.3.6) coincides with the $\mathfrak{g}$-action it inherits as a submodule of $\mathcal{PD}(W)$ (Lemma 3.3.9).*

Proof. Let $x \in \mathfrak{g}$, $w \in W$ and $v^* \in W^*$ be homogeneous. Then

$$\begin{aligned} \left(\pi_{\mathcal{D}}(x)\partial_w\right)(v^*) &= \partial_{\pi(x)w}(v^*) && \text{Lemma 3.3.6} \\ &= (-1)^{|v^*|(|x|+|w|)}\langle v^*, \pi(x)w\rangle && \text{Equation (3.3.1)} \\ &= \left(\pi_{\tilde{\mathcal{D}}}(x)\partial_w\right)(v^*). && \text{Lemma (3.3.9)} \end{aligned}$$

which proves that $\pi_{\mathcal{D}}(x)\partial_w$ and $\pi_{\tilde{\mathcal{D}}}(x)\partial_w$ agree on the generators of $\mathcal{P}(W)$. Then we assert that $\pi_{\mathcal{D}}(x)\partial_w$ and $\pi_{\tilde{\mathcal{D}}}(x)\partial_w$ agree on $\mathcal{P}(W)$. More precisely, we notice that $\pi_{\mathcal{D}}(x)\partial_w = \partial_{\pi(x)w}$ is a derivation. Also

$$\pi_{\tilde{\mathcal{D}}}(x)\partial_w = \pi^*(x)\partial_w - (-1)^{|x||w|}\partial_w\pi^*(x) = [\pi^*(x), \partial_w]$$

is a derivation as well by the fact that if D_1, D_2 are derivations on a superalgebra A, then so is $[D_1, D_2]$. Then the assertion follows from the fact that if two derivations agree on generators of a superalgebra, they agree everywhere. To complete the proof, we need to show that $\pi_{\mathcal{D}}(x)\partial_b = \pi_{\tilde{\mathcal{D}}}(x)\partial_b$ for arbitrary $b \in \mathcal{S}(W)$. Thus by Definition 3.3.4, it suffices to show that for homogenous $w_1, \ldots, w_n \in W$, we have

$$\begin{aligned}\pi_{\mathcal{D}}(x)\left(\partial_{w_1}\ldots\partial_{w_n}\right) &= \pi^*(x)\partial_{w_1}\ldots\partial_{w_n} - (-1)^{|x|\sum_{i=1}^n |w_i|}\partial_{w_1}\ldots\partial_{w_n}\pi^*(x) \\ &= \pi_{\tilde{\mathcal{D}}}(x)\left(\partial_{w_1}\ldots\partial_{w_n}\right),\end{aligned}$$

where the second equality follows from Equation (3.2.2) and the fact the the parity of $\partial_{w_1}\ldots\partial_{w_n}$ is $\sum_{i=1}^n |w_i|$. We proceed the proof by induction on n. The base case when $n=1$ is done by the first step of the proof of this lemma. Suppose that the result holds for all $r < n$. We have

$$\begin{aligned}\pi_{\mathcal{D}}(x)\left(\partial_{w_1}\ldots\partial_{w_n}\right) &= \left(\pi_{\mathcal{D}}(x)\partial_{w_1}\right)\left(\partial_{w_2}\ldots\partial_{w_n}\right) + (-1)^{|x||w_1|}\partial_{w_1}\left(\pi_{\mathcal{D}}(x)\left(\partial_{w_2}\ldots\partial_{w_n}\right)\right) \\ &= \left(\pi_{\tilde{D}}(x)\partial_{w_1}\right)\left(\partial_{w_2}\ldots\partial_{w_n}\right) + (-1)^{|x||w_1|}\partial_{w_1}\left(\pi_{\tilde{D}}\left(\partial_{w_2}\ldots\partial_{w_n}\right)\right)\end{aligned}$$

which is

$$\begin{aligned}&\left(\pi^*(x)\partial_{w_1} - (-1)^{|x||w|}\partial_{w_1}\pi^*(x)\right)\left(\partial_{w_2}\ldots\partial_{w_n}\right) \\ &\qquad + (-1)^{|x||w_1|}\partial_{w_1}\left(\pi^*(x)\partial_{w_2}\ldots\partial_{w_n} - (-1)^{|x|\sum_{i=2}^n |w_i|}\partial_{w_2}\ldots\partial_{w_n}\pi^*(x)\right)\end{aligned}$$

which can by simplified to

$$\pi^*(x)\partial_{w_1}\ldots\partial_{w_n} - (-1)^{|x|\sum_{i=1}^n |w_i|}\partial_{w_1}\ldots\partial_{w_n}\pi^*(x)$$

as claimed. □

Lemma 3.3.11. *The map*

$$\mathtt{m} : \mathcal{P}(W) \otimes \mathcal{S}(W) \to \mathcal{PD}(W) \text{ such that } p \otimes s \mapsto p\partial_s$$

gives an isomorphism between $\mathcal{PD}(W)$ *and* $\mathcal{P}(W) \otimes \mathcal{S}(W)$ *as* $\mathfrak{g}$*-modules.*

Proof. It is suffices to only consider homogenous elements. Let $x \in \mathfrak{g}$, $p \in \mathcal{P}(W)$ and $s \in \mathcal{S}(W)$ be homogenous. Recall that the action of $\mathfrak{g}$ on $\mathcal{PD}(W)$ is given by Equation (3.2.2). Then for arbitrary $f \in \mathcal{P}(W)$, we have that

$$\begin{aligned}
\mathtt{m}\left(x \cdot (p \otimes s)\right)(f) &= \mathtt{m}\left(\pi^*(x)p \otimes s + (-1)^{|p||x|} p \otimes \pi(x)s\right)(f) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|p||x|}\left(p\partial_{\pi(x)s}(f)\right) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|p||x|} p\left(\pi_{\mathcal{D}}(x)\partial_s\right)(f) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|p||x|} p\left(\pi_{\tilde{\mathcal{D}}}(x)\partial_s\right)(f) \qquad \text{by Proposition 3.3.10.}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
x \cdot \left(\mathtt{m}\left(p \otimes u\right)\right)(f) &= \pi^*(x)\left(\left(p\partial_s\right)(f)\right) - (-1)^{|x|(|p|+|s|)}\left(p\partial_s\right)\left(\pi^*(x)f\right) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|x||p|} p\pi^*(x)\left(\partial_s f\right) - (-1)^{|x|(|p|+|s|)}\left(p\partial_s\right)\left(\pi^*(x)f\right) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|x||p|}\Big(p\pi^*(x)\left(\partial_s f\right) - (-1)^{|x||s|}\left(p\partial_s\right)\left(\pi^*(x)f\right)\Big) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|x||p|} p\Big(\pi^*(x)\left(\partial_s f\right) - (-1)^{|x||s|}\left(\partial_s\right)\left(\pi^*(x)f\right)\Big) \\
&= \left(\pi^*(x)p\right)\partial_s(f) + (-1)^{|x||p|} p\left(\pi_{\tilde{\mathcal{D}}}(x)\partial_s\right)(f).
\end{aligned}$$

which completes the proof. □

Remark 3.3.12. The map $\mathtt{m} : \mathcal{P}(W) \otimes \mathcal{S}(W) \to \mathcal{PD}(W)$ is an isomorphism of $\mathfrak{g}$-modules, but not of superalgebras as $\mathcal{PD}(W)$ is not (super) commutative. Let $g \in \mathcal{P}(W)_{\bar{0}}$ and $x \in W_{\bar{0}}$. Let $f(x) = x$. Then

$$\partial_x(fg) = \partial_x(f)g + (-1)^{|x||f|} f\partial_x(g) = g + f\partial_x(g)$$

by the fact $f = x$ and $|x| = |f| = 0$. Thus $\partial_x f = 1 + f\partial_x$ which implies $\partial_x f - f\partial_x = 1$. However, $\mathcal{P}(W) \otimes \mathcal{S}(W)$ is (super) commutative algbera.

3.4 The Capelli Eigenvalue Problem

In this section, we define the Capelli Eigenvalue Problem. Let $\mathfrak{g}$ be a Lie superalgebra and W a $\mathfrak{g}$-module such that $\mathcal{S}(W)$ has a multiplicity-free decomposition which is parametrized by a set Ω, that is,

$$\mathcal{S}(W) \cong \bigoplus_{\lambda \in \Omega} W_\lambda, \tag{3.4.1}$$

where the W_λ's are pairwise non-isomorphic irreducible $\mathfrak{g}$-modules such that the space $\mathrm{Hom}_{\mathfrak{g}}(W_\lambda, W_\lambda)$ is 1-dimensional. We are interested in the (super)algebra $\mathcal{PD}(W)^{\mathfrak{g}}$, the algebra of $\mathfrak{g}$-invariant polynomial-coefficient differential operators.

Lemma 3.4.1. *As a $\mathfrak{g}$-module, we have that*

$$\mathcal{PD}(W)^{\mathfrak{g}} \cong \bigoplus_{\lambda \in \Omega} \mathbb{C} id_{W_\lambda} \tag{3.4.2}$$

Proof. Recall that from Lemma 3.3.11, $\mathcal{PD}(W) \cong \mathcal{P}(W) \otimes \mathcal{S}(W)$ as $\mathfrak{g}$-modules. By

Equation (3.4.1) and the fact that $\mathcal{P}(W) \cong \mathcal{S}(W^*)$ as $\mathfrak{g}$-modules, we have that

$$\begin{aligned}
\mathcal{PD}(W)^{\mathfrak{g}} \cong (\mathcal{P}(W) \otimes \mathcal{S}(W))^{\mathfrak{g}} &\cong (\mathcal{P}(W) \otimes \mathcal{P}(W^*))^{\mathfrak{g}} \\
&\cong \left(\bigoplus_{\lambda \in \Omega} W_\lambda \otimes \bigoplus_{\mu \in \Omega} W_\mu^* \right)^{\mathfrak{g}} \\
&\cong \bigoplus_{\lambda, \mu \in \Omega} \left(W_\lambda \otimes W_\mu^* \right)^{\mathfrak{g}} \\
&\cong \bigoplus_{\lambda, \mu \in \Omega} \mathrm{Hom}_{\mathfrak{g}} \left(W_\mu, W_\lambda \right) \text{ by Lemma 3.1.1 and 3.2.3} \\
&\cong \bigoplus_{\lambda \in \Omega} \mathrm{Hom}_{\mathfrak{g}}(W_\lambda, W_\lambda) \\
&\cong \bigoplus_{\lambda \in \Omega} \mathbb{C} id_{W_\lambda}
\end{aligned}$$

where the last two isomorphisms follow from Schur's lemma [CW13, Lemma 3.4]. □

Definition 3.4.2. *The Capelli operator* D^λ *for* $\lambda \in \Omega$ *is the* $\mathfrak{g}$*-invariant differential operator in* $\mathcal{PD}(W)^{\mathfrak{g}}$ *that corresponds to* id_{W_λ} *in* $\mathrm{Hom}_{\mathfrak{g}}(W_\lambda, W_\lambda)$ *via the isomorphism* (3.4.2).

Lemma 3.4.3. *The set* $\{D^\lambda \mid \lambda \in \Omega\}$ *forms a basis for* $\mathcal{PD}(W)^{\mathfrak{g}}$.

Proof. The result follows from Lemma 3.4.1. □

Remark 3.4.4. From Lemma 3.4.1, it might look like the Capelli operator D_μ acts by 1 on W_μ and by 0 on the other W_λ's. However, this is not the case. The source for this confusion might be the embedding

$$\bigoplus_{\lambda, \mu \in \Omega} \mathrm{Hom}(W_\mu, W_\lambda) \subseteq \mathrm{Hom} \left(\bigoplus_{\mu \in \Omega} W_\mu, \bigoplus_{\lambda \in \Omega} W_\lambda \right) \subseteq \mathrm{End}(\mathcal{P}(W)). \tag{3.4.3}$$

But the algebra $\mathcal{PD}(W)$ does not embed into $\mathrm{End}(\mathcal{P}(W))$ under these isomorphisms.

For example $\mathcal{PD}(W)$ is not isomorphic to $\mathcal{P}(W)\otimes\mathcal{D}(W)$ as algebras by Remark 3.3.12. Thus the action of the element id_{W_μ} on $\mathcal{P}(W)$ arising from the identification (3.4.3) is not the same as the action of the Capelli operator D^μ on $\mathcal{P}(W)$. By $\mathfrak{g}$-invariance, D^μ acts by scalars on each irreducible $\mathfrak{g}$-module W_μ, as does each element of $\mathrm{End}(\mathcal{P}(W))^{\mathfrak{g}}$, but these scalars are not directly related. For example, id_{W_μ} acts by 0 on all but finitely many components, but no differential operator could act this way.

Example 3.4.5. Let $\mathfrak{g}=\mathfrak{gl}_2(\mathbb{C})\times\mathfrak{gl}_2(\mathbb{C})$. Let $V=\mathbb{C}^2\otimes(\mathbb{C}^2)^*$ be the natural $\mathfrak{g}$-module of $\mathfrak{g}$, such that $\mathbb{C}^2$ is the standard $\mathfrak{gl}_2(\mathbb{C})$-module and $(\mathbb{C}^2)^*$ is the contragradient module of $\mathbb{C}^2$. The by [CW01, Theorem 3.2], $\mathcal{P}(V)$ has a multiplicity-free decomposition as a $\mathfrak{g}$-module. By Lemma 3.4.1, we have that

$$\begin{aligned}\mathcal{PD}(V)^{\mathfrak{g}} &\cong (\mathcal{P}(V)\otimes\mathcal{S}(V))^{\mathfrak{g}}\\ &\cong \bigoplus_{i,j}^{\infty}\left(\mathcal{P}^i(V)\otimes\mathcal{P}^j(V^*)\right)^{\mathfrak{g}}\\ &\cong \left(\mathcal{P}^0(V)\otimes\mathcal{P}^0(V^*)\right)^{\mathfrak{g}}\oplus\left(\mathcal{P}^1(V)\otimes\mathcal{P}^1(V^*)\right)^{\mathfrak{g}}\oplus\cdots\\ &\cong \mathbb{C}\oplus(V\otimes V^*)^{\mathfrak{g}}\oplus\cdots.\end{aligned}$$

Now we find the Capelli operator D^λ such that λ corresponds to $(V\otimes V^*)^{\mathfrak{g}}$. Let $\{e_1,e_2\}$ be the standard basis of $\mathbb{C}^2$. Let $\{e_1^*,e_2^*\}$, $\{v_{i,j}=e_i\otimes e_j^*\}_{1\le i,j\le 2}$ and $\{v_{i,j}^*=e_i^*\otimes e_j\}_{1\le i,j\le 2}$ be basis for $(\mathbb{C}^2)^*$, V and V^* respectively. Recall that $\mathcal{P}^1(V)$ is the ring of polynomials of four indeterminates ($\{v_{i,j}\}_{1\le i,j\le 2}$) of homogenous degree 1. Moreover, we have that $\partial_{v_{i,j}}(v_{k,\ell})=\delta_{i,k}\delta_{j\ell}$. Since a $\mathfrak{g}$-invariant vector X in $V^*\otimes V$ is of the form $X=\sum_{1\le i,j\le 2}v_{i,j}^*\otimes v_{i,j}$, we may identify X with $D^\lambda=\sum_{1\le i,j\le 2}v_{i,j}\partial_{v_{i,j}}$, which clearly belongs to $\mathcal{PD}(V)$, and by Lemma 3.4.1, $D^\lambda\in\mathcal{PD}(V)^{\mathfrak{g}}$. Note that D^λ acts on $\mathcal{P}^d(V)$ by the degree d. ♠

By Definition 3.4.3, the set of Capelli operators $\{D^\mu \mid \mu \in \Omega\}$ forms a basis for $\mathcal{PD}(W)^{\mathfrak{g}}$. Then each Capelli operator D^μ acts on each irreducible component $W_\lambda \subset \mathcal{S}(W)$ by a scalar eigenvalue $c_\mu(\lambda)$. This allows us to define the *Capelli Eigenvalue Problem.*

The Capelli Eigenvalue Problem. Find the eigenvalue $c_\mu(\lambda)$ for each $\lambda, \mu \in \Omega$.

We finish this chapter by clarifying the relation between previous work on the CEP for Lie superalgebras and the work that is done in this thesis. The Capelli Eigenvalue problem has been solved in many cases, including $\left(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)$ and $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$. In[SSS20], the authors first parametrize the representations of $\mathfrak{g}$ by hook partitions. Then they fix a Borel subalgebra $\mathfrak{b}$ (coming from the standard Borel subalgebra or its opposite), and write down a formula for obtaining the $\mathfrak{b}$-highest weights of irreducible components of $\mathcal{P}(V)$ from the hook partition. Finally they show that the formula for the eigenvalue $c_\mu(\lambda)$ can be computed as a polynomial in the $\mathfrak{b}$-highest weight. The fixed choice of the Borel subalgebra (see [SSS20, Table 4]) essentially means that the formula for the eigenvalues of a Capelli operator is really dependent on the parametrization of modules by partitions.

Our goal in this thesis is to solve the following problem for the particular pairs $\left(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)$ and $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$.

The Refined Capelli Eigenvalue Problem. For any Borel subalgebra $\mathfrak{b}$, find the eigenvalue $c_\mu(\lambda)$ as a polynomial function in the $\mathfrak{b}$-highest weight of W_λ.

This is a more desirable solution to the CEP, and as we shall see, not always possible. In the next chapter, we describe the interpolation polynomials that are the key to the solution.

Chapter 4

Super Symmetric Polynomials

In this chapter, we give the definition of the interpolation super Jack polynomials, which turn out to be the polynomials that arise in the solution to the Capelli Eigenvalue Problem which was introduced in Section 3.4. We first define monomial symmetric polynomials and power sum polynomials which will be used to define Jack polynomials. Then we define an inhomogeneous variation of Jack polynomials called interpolation Jack polynomials whose leading terms are precisely Jack polynomials. We extend this class of polynomials by defining the interpolation super Jack polynomials, which are the super-analogues of interpolation Jack polynomials. For further details on these polynomials, we refer the reader to [Mac95], [KS96] and [SV05].

4.1 Symmetric Polynomials

Let $\mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n indeterminates with coefficients in $\mathbb{C}$. The symmetric group $\mathcal{S}_n$ acts on $\mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables. Let $\Lambda_n \coloneqq \mathbb{C}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ be the subring of polynomials fixed by the action of the symmetric group $\mathcal{S}_n$, that is, the ring of symmetric polynomials in n variables. Then

$\Lambda_n = \bigoplus_{r\geq 0} \Lambda_n^r$ is a graded ring where Λ_n^r consists of the homogeneous symmetric polynomials of degree r. For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We next give two commonly used symmetric polynomials, which also are two common bases for Λ_n. For more details of symmetric polynomials, we refer the readers to [Mac95].

Definition 4.1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition such that $\ell(\lambda) \leq n$. The* monomial symmetric polynomial m_λ *is defined as*

$$m_\lambda (x_1, \ldots, x_n) \coloneqq \sum_\alpha x^\alpha \tag{4.1.1}$$

where α ranges over all distinct permutations of the n-tuple formed by the parts of λ.

Example 4.1.2. Consider the symmetric monomial polynomials $m_\lambda(x_1, x_2)$ in two indeterminates and $n = 2$. First notice that $m_\varnothing(x_1, x_2) = 1$ for the empty partition $\varnothing$. We also have

$$m_{(1,0)}(x_1, x_2) = x_1 + x_2,\ m_{(1,1)}(x_1, x_2) = x_1x_2,\ m_{(2,0)}(x_1, x_2) = x_1^2 + x_2^2. \qquad \spadesuit$$

Definition 4.1.3. *For each positive integer r and any number of indeterminates x_i, the r-th* power sum polynomial *is defined as*

$$p_r = \sum x_i^r = m_{(r)}.$$

Moreover, for a partition $\lambda = (\lambda_1, \dots, \lambda_m)$*, the* power sum polynomial p_λ *is defined by*

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_m}.$$

Example 4.1.4. Consider the power sum polynomials $p_\lambda(x_1, x_2)$ in two indeterminates. First notice that $p_\varnothing(x_1, x_2) = 1$ for the empty partition $\varnothing$. We also have

$$p_{(1)}(x_1, x_2) = x_1 + x_2,\ p_{(1,1)}(x_1, x_2) = (x_1 + x_2)^2,\ p_{(2)}(x_1, x_2) = x_1^2 + x_2^2. \qquad \spadesuit$$

Remark 4.1.5. In fact, $\{m_\lambda\}_{\lambda \in \mathcal{P}_n, |\lambda| = r}$ forms a basis of Λ_n^r. Thus the set $\{m_\lambda\}_{\lambda \in \mathcal{P}_n}$ forms a graded basis for Λ_n. Similarly, $\{p_\lambda\}_{\lambda \in \mathcal{P}_n}$ also forms a graded basis for Λ_n. See [Mac95, Section VI].

4.2 Definition of Jack Polynomials

In this section, we define Jack polynomials by introducing a scalar product on Λ_n. Let us begin by defining a positive integer associated with each partition, as follows. For any partition λ, define

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! \tag{4.2.1}$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. For example, we have

$$z_{(1,1)} = 1^2 2! = 2 \text{ and } z_{(2)} = 2^1 1! = 2. \tag{4.2.2}$$

Let $\mathbb{C}(\theta)$ be the set of rational functions in one variable. We can define an $\mathbb{C}(\theta)$-valued scalar product $\langle \cdot, \cdot \rangle_\theta$ on Λ_n by requiring that the power sum polynomials are an orthogonal basis with respect to $\langle \cdot, \cdot \rangle_\theta$ and such that

$$\langle p_\lambda, p_\mu \rangle_\theta = (1/\theta)^{\ell(\lambda)} \delta_{\lambda\mu} z_\lambda. \tag{4.2.3}$$

Then we have the following definition.

Definition 4.2.1. *[Mac95, Section VI.4]Let $\{\mathbf{J}_\lambda\}_{\lambda \in \mathcal{P}_n}$ denote the set of symmetric polynomials obtained by applying the Gram-Schmidt algorithm to the ordered basis $\{m_\lambda\}_{\lambda \in \mathcal{P}_n}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_\theta$. We call the $\mathbf{J}_\lambda$ Jack polynomials.*

Example 4.2.2. We compute the Jack polynomials in two indeterminates x_1, x_2. For simplicity, we drop (x_1, x_2) in the following computation. Recall that from 4.1.1 and 4.1.3, we have that

$$\begin{aligned} m_\varnothing &= p_\varnothing, & m_{(1)} &= p_{(1)} \\ m_{(1,1)} &= \frac{1}{2}\left(p_{(1,1)} - p_{(2)}\right), & m_{(2)} &= p_{(2)}. \end{aligned}$$

Let $\mathbf{J}_\varnothing = m_\varnothing$. Since $m_\phi, m_{(1)}$ and $m_{(1,1)}$ are already orthogonal, we have $\mathbf{J}_{(1)} = m_{(1)}$ and $\mathbf{J}_{(1,1)} = m_{(1,1)}$ but

$$\begin{aligned} \mathbf{J}_{(2)} &= m_{(2)} - \frac{\langle m_{(1,1)}, m_{(2)} \rangle_\theta}{\langle m_{(1,1)}, m_{(1,1)} \rangle_\theta} m_{(1,1)} \\ &= p_{(2)} - \frac{\frac{1}{2}\langle p_{(1,1)} - p_{(2)}, p_{(2)} \rangle_\theta}{\frac{1}{4}\langle p_{(1,1)} - p_{(2)}, p_{(1,1)} - p_{(2)} \rangle_\theta} \frac{1}{2}\left(p_{(1,1)} - p_{(2)}\right). \end{aligned}$$

By Equations (4.2.2) and (4.2.3), the numerator becomes

$$\frac{1}{2}\langle -p_{(2)}, p_{(2)} \rangle_\theta = \frac{1}{2}(-1)2\left(\frac{1}{\theta}\right)^1 = -\frac{1}{\theta}$$

and the denominator becomes

$$\frac{1}{4}\left(\langle p_{(1,1)}, p_{(1,1)}\rangle_\theta + \langle p_{(2)}, p_{(2)}\rangle_\theta\right) = \frac{1}{4}\left(2\left(\frac{1}{\theta}\right)^2 + 2\left(\frac{1}{\theta}\right)\right).$$

Thus by simplification, we have

$$\mathbf{J}_{(2)} = p_{(2)} - \frac{-\left(\frac{1}{\theta}\right)}{\frac{1}{4}\left(2\left(\frac{1}{\theta}\right)^2 + 2\left(\frac{1}{\theta}\right)\right)}\frac{1}{2}\left(p_{(1,1)} - p_{(2)}\right) = x_1^2 + x_2^2 + \frac{2\theta}{\theta+1}x_1x_2$$

which is

$$m_{(2)} + \frac{2\theta}{\theta+1}m_{(1,1)}.$$

♠

By Definition 4.2.1, the Jack polynomials are obtained from an ordered basis of the algebra of symmetric polynomials by Gram-Schmidt algorithm. It follows there exists a strictly upper unitriangular transition matrix that expresses the Jack polynomial in terms of monomial symmetric functions. That is, $\mathbf{J}_\lambda$ is of the form

$$\mathbf{J}_\lambda = m_\lambda + \sum_{\mu \prec \lambda} c_{\lambda,\mu} m_\mu \tag{4.2.4}$$

where $c_{\mu,\lambda} \in \mathbb{C}$ and $\prec$ is the partial order defined in Definition 2.2.7. Even better, as shown in [Sta89, Theorem 1.1], the transition matrix is upper unitrianguler with respect to the coarser partial order $<$ defined in Definition 2.2.8. That is, we can replace $\mu \prec \lambda$ by $\mu < \lambda$ in Equation (4.2.4) .

In the next section, we give the definition of interpolation Jack polynomials whose leading terms are Jack polynomials.

4.3 Interpolation Jack Polynomials

In this section we define the interpolation Jack polynomials. Let θ be a parameter. Let

$$\rho = \theta(n-1, \dots, 1, 0) \in \mathbb{C}^n.$$

Recall that a partition can be represented by its associated Young diagram. Thus, we may identify each box $(i,j) \in \mathbb{Z}^2$ with $1 \le i \le n$ and $1 \le j \le \lambda_i$ in a partition. We say $s \in \lambda$ if s is a box in the associated Young diagram of λ. For each box $s = (i,j) \in \lambda$, we define

$$\begin{aligned} c_\lambda^\rho(s) &:= (\lambda_i - j + 1) + (\rho_i - \rho_{\lambda_j'}) \\ &= (\lambda_i - j + 1) + (\theta(n-i) - \theta\left(n - \lambda_j'\right)) \\ &= (\lambda_i - j + 1) + \theta\left(\lambda_j' - i\right). \end{aligned}$$

Therefore, $c_\lambda^\rho(s)$ equals the number of boxes in the blue rectangle plus θ times the number of boxes in the red rectangle, as indicated in Figure 4.1. When $\theta = 1$, the formula is known as the *hook length* of box (i,j).

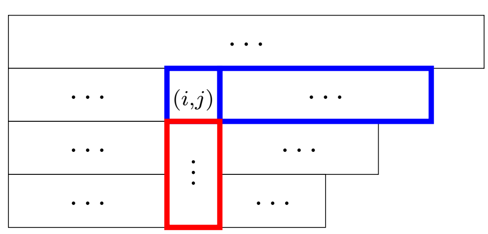


Figure 4.1: $c_\lambda^\rho((i,j))$

Example 4.3.1. Consider the partition $\lambda = (2,0)$ and $\rho = \theta(1,0)$. Then from Figure 4.2, we derive $c_\lambda^\rho(s_1) = 2$ and $c_\lambda^\rho(s_2) = 1$ as follows.

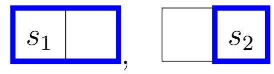


Figure 4.2: The value of $c_\lambda^\rho(s_1)$ and $c_\lambda^\rho(s_2)$.

♠

Definition 4.3.2. *([KS96] Section 2) For any partition λ such that $\ell(\lambda) \leq n$, the interpolation Jack polynomial $\mathbf{P}_\lambda^\rho$ is the unique polynomial in n variables with coefficients in $\mathbb{C}(\theta)$, which is characterized by the following properties*

(i) $\mathbf{P}_\lambda^\rho \in \Lambda_n$;

(ii) $deg(\mathbf{P}_\lambda^\rho) \leq |\lambda|$;

(iii) $\mathbf{P}_\lambda^\rho(\lambda + \rho) = \prod_{s \in \lambda} c_\lambda^\rho(s)$.

(iv) $\mathbf{P}_\lambda^\rho(\mu + \rho) = 0$ *for all partitions* μ *such that* $\ell(\mu) \leq n$, $|\mu| \leq |\lambda|$ *and* $\mu \neq \lambda$.

Example 4.3.3. Let $\lambda = (2, 0)$ and $\rho = \theta(1, 0)$. We compute the corresponding interpolation Jack polynomial of two variables. First notice that by Definition 4.3.2 (ii) we must have

$$\mathbf{P}_\lambda^\rho(x_1, x_2) = a(x_1^2 + x_2^2) + b(x_1 x_2) + c(x_1 + x_2) + d$$

for some $a, b, c, d \in \mathbb{R}$. Then by Definition 4.3.2 (iii) and Example 4.3.1, we have

$$\mathbf{P}_\lambda^\rho\left(\lambda + \rho\right) = \mathbf{P}_\lambda^\rho\left(2 + \theta, 0\right) = a\left(2 + \theta\right)^2 + c\left(2 + \theta\right) + d = 2. \tag{4.3.1}$$

Now consider $\mu^1 \prec \mu^2 \prec \mu^3 \prec \lambda$ where $\mu^1 = (0, 0)$, $\mu^2 = (1, 0)$ and $\mu^3 = (1, 1)$. We have $\mathbf{P}^{\rho}_{\lambda}(\mu^i+\rho)=0$ for all $i=1,2,3$ by 4.3.2 (iv). Thus we have that

$$\mathbf{P}^{\rho}_{\lambda}\left(\mu^1+\rho\right)=a\left(\theta\right)^2+c\left(\theta\right)+d=0 \tag{4.3.2}$$

$$\mathbf{P}^{\rho}_{\lambda}\left(\mu^2+\rho\right)=a\left(1+\theta\right)^2+c\left(1+\theta\right)+d=0, \tag{4.3.3}$$

$$\mathbf{P}^{\rho}_{\lambda}\left(\mu^1+\rho\right)=a\left(\left(1+\theta\right)^2+1\right)+b\left(1+\theta\right)+c\left(2+\theta\right)+d=0. \tag{4.3.4}$$

After solving the linear system formed by Equations (4.3.1)-(4.3.4), we have that

$$a=1, \qquad b=\frac{2\theta}{\theta+1}, \qquad c=-1-2\theta, \qquad d=\theta^2+\theta$$

Therefore, we obtain

$$\mathbf{P}^{(\theta,0)}_{(2,0)}(x_1,x_2)=\underbrace{(x_1^2+x_2^2)+\frac{2\theta}{\theta+1}x_1x_2}_{\mathbf{J}_{(2)}}+\left(-1-2\theta\right)\left(x_1+x_2\right)+\theta^2+\theta,$$

whose top degree homogeneous part is exactly $\mathbf{J}_{(2)}$ as we computed in Example 4.3.3. Similarly, we can compute $\mathbf{P}_{(0,0)}=1$, $\mathbf{P}_{(1,0)}=\mathbf{J}_{(1,0)}$ and $\mathbf{P}_{(1,1)}=\mathbf{J}_{(1,1)}$. ♠

Here we quote a result from [KS96] which reveals the relation between Jack polynomials and interpolation Jack polynomials.

Theorem 4.3.4 ([KS96] Corollary 4.7)**.** *Let λ be a partition such that $\ell(\lambda)\leq n$. Let $\rho=\theta(n-1,n-2,\dots,0)$ where θ is a parameter. The top homogeneous part of the interpolation Jack polynomial $\mathbf{P}^{\rho}_{\lambda}$ is precisely the Jack polynomial $\mathbf{J}_{\lambda}$.*

Remark 4.3.5. In fact in the paper [KS96], F. Knop and S. Sahi defined a larger class of polynomials called the interpolation polynomials where $\rho=(\rho_1,\dots,\rho_n)$ is an arbitrary vector in $\mathbb{C}^n$ satisfying only the condition $\rho_i-\rho_j\neq 1,2,3,\dots$ for all

$1 \leq i < j \leq n$. The interpolation Jack polynomial is just a special case of the interpolation polynomial.

In [KS96], Sahi proved that the solution to the Capelli eigenvalue problem can be interpolated by a polynomial in the highest weights that is characterized uniquely by certain symmetry, vanishing and degree conditions. Knop and Sahi later showed in a wide range of cases that the top degree component of such polynomials were the celebrated Jack polynomials.

Recently, S. Sahi, H. Salmasian and V. Serganova proved in [SSS20] that an analogous result holds in the setting of Lie superalgebra by using the *Sergeev-Veselov's interpolation super Jack polynomials*.

We finish this chapter by giving their definition as in [SSS20, Theorem 1.8], and we shall discuss more about the result in [SSS20] in the next chapters where we find a refined solution to the Capelli Eigenvalue Problem.

4.4 Interpolation super Jack Polynomials

In this section, let $m, n \in \mathbb{Z}_{\geq 0}$. Let $\mathbb{C}[x_1, \ldots, x_m | y_1 \ldots, y_n]$ be the ring of polynomials in $m+n$ indeterminates with coefficients in $\mathbb{C}$. Let $\theta \in \mathbb{C}$. We say a polynomial $f(x_1, \ldots, x_m | y_1, \ldots, y_n) \in \mathbb{C}[x_1, \ldots, x_m | y_1, \ldots, y_n]$ is *separately symmetric* if f is symmetric on $\{x_i\}_{1 \leq i \leq m}$ and on $\{y_j\}_{1 \leq j \leq n}$ separately. We first define an important class of separately symmetric polynomials.

Definition 4.4.1. *We denote by $\Lambda_{m,n,\theta}$ the subalgebra of separately symmetric polynomials f such that*

$$f\left(\ldots, x_i + \frac{1}{2}, \ldots, \ldots, y_j - \frac{1}{2}, \ldots\right) = f\left(\ldots, x_i - \frac{1}{2}, \ldots, \ldots, y_j + \frac{1}{2}, \ldots\right) \quad (4.4.1)$$

on every hyperplane $x_i + \theta y_j = 0$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*. We call the symmetry property defined in Equation* (4.4.1) *monoidal symmetry.*

Definition 4.4.2. *Let* $u, v \in \mathbb{C}^{m|n}$*. We say* u *is equivalent to* v *if* $f(u) = f(v)$ *for all* $f \in \Lambda_{m,n,\theta}$*. We write* $u \sim v$ *if* u *is equivalent to* v*.*

Lemma 4.4.3. *Let* $(x|y) = (x_1, \ldots, x_m|y_1, \ldots, y_n) \in \mathbb{C}^{m|n}$*. We have that*

$$(x|y) \sim (x - e_i|y + e_{m+j})$$

whenever $x_i + \theta y_j = \frac{1}{2}(1 - \theta)$*, and*

$$(x|y) \sim (x + e_i|y - e_{m+j})$$

whenever $x_i + \theta y_j = -\frac{1}{2}(1 - \theta)$*.*

Proof. We have that $(u + \frac{1}{2}e_i|v - \frac{1}{2}e_{m+j}) \sim (u - \frac{1}{2}e_i|v + \frac{1}{2}e_{m+j})$ whenever $u_i + \theta v_j = 0$. Setting $x = u + \frac{1}{2}e_i$ and $y = v - \frac{1}{2}e_{m+j}$ yields that $x_i + \theta y_j = \frac{1}{2}(1 - \theta)$ if and only if $u_i + \theta v_j = 0$, in which case the first condition holds. Alternately, setting $x = u - \frac{1}{2}e_i$, $y = v + \frac{1}{2}e_{m+j}$, we have $x_i + \theta y_j = -\frac{1}{2}(1 - \theta)$ if and only if $u_i + \theta v_j = 0$, in which case the second condition holds. □

We need one extra definition before we can define the interpolation super Jack polynomials. Given an (m, n)-hook partition $\lambda \in \mathcal{H}(m, n)$ (Definition 2.2.11), the *twisted Frobenius coordinates* $(p(\lambda), q(\lambda)) := (p_1(\lambda), \ldots, p_m(\lambda), q_1(\lambda), \ldots, q_n(\lambda))$ of λ are defined as follows:

$$\begin{aligned} p_i(\lambda) &:= \lambda_i - \theta\left(i - \frac{1}{2}\right) - \frac{1}{2}\left(n - \theta m\right), \text{ and} \\ q_j(\lambda) &:= \langle \lambda'_j - m\rangle - \theta^{-1}\left(j - \frac{1}{2}\right) + \frac{1}{2}\left(\theta^{-1} n + m\right), \end{aligned} \tag{4.4.2}$$

where $1 \leq i \leq m, 1 \leq j \leq n$ and $\langle x \rangle := \max\{0, x\}$ for all $x \in \mathbb{R}$.

Definition 4.4.4. *([SV05] Section 6)Let λ, μ be (m,n)-hook partitions. The interpolation super Jack polynomials $\mathbf{SP}^*_\lambda$ are polynomials in $m+n$ variables with coefficients in $\mathbb{C}(\theta)$, that are uniquely determined by the following properties:*

(i) $\mathbf{SP}^*_\lambda \in \Lambda_{m,n,\theta}$;

(ii) *deg*$(\mathbf{SP}^*_\lambda) \leq |\lambda|$ *where the degree of* $\mathbf{SP}^*_\lambda$ *is the total degree of both x and y;*

(iii) $\mathbf{SP}^*_\lambda(p(\lambda), q(\lambda); \theta) = \prod_{s \in \lambda} c^\rho_\lambda(s)$;

(iv) $\mathbf{SP}^*_\lambda(p(\mu), q(\mu); \theta) = 0$ *for all (m,n)-hook partitions μ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.*

Furthermore, the set $\{\mathbf{SP}^*_\lambda\}_{\lambda \in \mathcal{H}(m,n)}$ forms a basis for $\Lambda_{m,n,\theta}$.

Remark 4.4.5. From [SSS20], it follows that by specializing θ to a value in $\mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, there are no poles in the coefficients of $\mathbf{SP}^*_\lambda$, and we obtain from $\mathbf{SP}^*_\lambda$ a polynomial $SP^*_{\lambda,\theta}$ with complex coefficients. Therefore, the polynomials $SP^*_{\lambda,\theta} \in \Lambda_{m,n,\theta}$ are polynomials in $m+n$ variables with complex coefficients, that are uniquely determined by the properties described in Definition 4.4.4. We also call $SP^*_{\lambda,\theta}$ interpolation super Jack polynomials.

In the next two chapters, we first summarize the results from [SSS20] for two cases: $\left(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)$ and $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$. In each case, they present a solution to the CEP in which the eigenvalues are computed as interpolation super Jack polynomials (with respectively, $\theta = 1, \frac{1}{2}$) evaluated on an affine function of the highest weights with respect to the standard (respectively, opposite standard) Borel subalgebra. In each case, we extend these formulae to provide solutions with respect to any Borel subalgebra.

Chapter 5

The CEP for $\left(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathbb{C}^{m|n} \otimes (\mathbb{C}^{m|n})^*\right)$

In this chapter, let $\mathfrak{g} := \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$ and $V := \mathbb{C}^{m|n} \otimes (\mathbb{C}^{m|n})^*$. Let $\mathfrak{h}_{m|n}$ be the standard Cartan subalgebra of $\mathfrak{gl}(m|n)$ with dual $\mathfrak{h}_{m|n}^*$. Then $\mathfrak{h} := \mathfrak{h}_{m|n} \oplus \mathfrak{h}_{m|n}$ is the standard Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}_{m|n}$ be the standard upper triangular Borel subalgebra of $\mathfrak{gl}(m|n)$ and $\mathfrak{b}_{m|n}^{\text{op}}$ the opposite standard Borel subalgebra of $\mathfrak{gl}(m|n)$. Then $\mathfrak{b}_{\text{st}} := \mathfrak{b}_{m|n}^{\text{op}} \oplus \mathfrak{b}_{m|n}$ is a Borel subalgebra of $\mathfrak{g}$. We begin with giving the solution to the Capelli Eigenvalue Problem for $(\mathfrak{g}, \mathfrak{b}_{\text{st}}, V)$ and then find a refined solution to the CEP for $(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n), \mathfrak{b}, V)$ for arbitrary $\mathfrak{b}$.

5.1 The Solution to the CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{st}}, V)$

To give the solution to the CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{st}}, V)$, we first show that $\mathcal{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module.

Theorem 5.1.1. *The polynomial algebra* $\mathcal{P}\left(\mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)$ *is completely reducible* *and multiplicity-free* $\mathfrak{g}$*-module, with the decomposition*

$$\mathcal{P}\left(\mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right) = \bigoplus_{\lambda \in \mathcal{H}(m|n)} \left(V_{m|n}^{\lambda}\right)^* \otimes V_{m|n}^{\lambda}.$$

In particular, for $\lambda = (\lambda_1, \ldots, \lambda_{m+n}) \in \mathcal{H}(m|n)$*, the highest weight* $\underline{\lambda}_{m|n}$ *of* $V_{m|n}^{\lambda}$ *with respect to* $\mathfrak{b}_{m|n}$ *is*

$$\sum_{i=1}^{m} \lambda_i \epsilon_i + \sum_{j=1}^{n} \langle \lambda_j' - m \rangle \delta_j$$

where $\langle x \rangle \coloneqq \max\{0, x\}$ *for all* x*. The highest weight of* $\left(V_{m|n}^{\lambda}\right)^*$ *with respect to* $\mathfrak{b}_{m|n}$ *is* $-\underline{\lambda}_{m|n}$*.*

Proof. Recall that from [CW01, Theorem 3.2], the symmetric algebra $\mathcal{S}\left(\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}\right)$ is completely reducible and multiplicity-free, with the decomposition

$$\mathcal{S}\left(\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}\right) = \bigoplus_{\lambda \in \mathcal{H}(m|n)} V_{m|n}^{\lambda} \otimes V_{m|n}^{\lambda},$$

where the highest weight of $V_{m|n}^{\lambda}$ is

$$\sum_{i=1}^{m} \lambda_i \epsilon_i + \sum_{j=1}^{n} \langle \lambda_j' - m \rangle \delta_j.$$

Further, given any $\mathfrak{gl}(m|n)$-module $E = V_{m|n}^{\lambda}$, we have $E^* \cong E^F$, where E^F is the $\mathfrak{gl}(m|n)$-module on which any $x \in \mathfrak{gl}(m|n)$ acts as $F(x)$ on E, where $F(x) = -x^{st}$, where *st* is the supertranspose. Then the result follows from the fact that

$$\mathcal{P}\left(\mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right) \cong \mathcal{S}\left(\left(\mathbb{C}^{m|n} \otimes \left(\mathbb{C}^{m|n}\right)^*\right)^*\right) \cong \mathcal{S}\left(\left(\mathbb{C}^{m|n}\right)^* \otimes \mathbb{C}^{m|n}\right). \qquad \square$$

We first define an affine map $\tau_0 : \mathfrak{h} \to \mathbb{C}^{m|n}$ before we give the solution to the

CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{st}}, V)$. First note that any element of $\mathfrak{h}^*_{m|n}$ has the form

$$\nu_{a,b} \coloneqq \sum_{i=1}^{m} a_i \epsilon_i + \sum_{j=1}^{n} b_j \delta_j,$$

for some $a \coloneqq (a_1, \dots, a_m) \in \mathbb{C}^m$ and $b \coloneqq (b_1, \dots, b_n) \in \mathbb{C}^n$.

Definition 5.1.2. *[SSS20, Table 3] For any $\lambda, \nu_{a,b} \in \mathfrak{h}^*_{m|n}$, the affine map $\tau_0 : \mathfrak{h}^* \to \mathbb{C}^{m|n}$ is defined by*

$$\tau_0((\lambda, \nu_{a,b})) \coloneqq \sum_{i=1}^{m} \left(a_i + \frac{m-n+1-2i}{2}\right) e_i + \sum_{j=1}^{n} \left(b_j + \frac{m+n+1-2j}{2}\right) e_{m+j}, \tag{5.1.1}$$

where $\{e_i, e_{m+j}\}_{1 \le i \le m, 1 \le j \le n}$ is the standard basis for $\mathbb{C}^{m|n}$.

That is, the affine map τ_0 only depends on the second entry. We finish this section by giving the solution to the CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{st}}, V)$ as follows.

Theorem 5.1.3. *[SSS20, Theorem 1.13.]Let $\mathfrak{g} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$, $\mathfrak{b}_{\text{st}} = \mathfrak{b}^{\text{op}}_{m|n} \oplus \mathfrak{b}_{m|n}$ and $V = \mathbb{C}^{m|n} \otimes (\mathbb{C}^{m|n})^*$. Then for each $\lambda, \mu \in \mathcal{H}(m|2n)$, the eigenvalue of the Capelli operator $D^{\mu}_{m|n}$ (Definition 3.4.2) on $\left(V^{\lambda}_{m|n}\right)^* \otimes V^{\lambda}_{m|n}$ with highest weight $\left(-\underline{\lambda}_{m|n}, \underline{\lambda}_{m|n}\right)$ with respect to $\mathfrak{b}_{\text{st}}$ is equal to*

$$SP^*_{\mu,1} \circ \tau_0 \left(\left(-\underline{\lambda}_{m|n}, \underline{\lambda}_{m|n}\right)\right),$$

where $SP^*_{\mu,1} \in \Lambda_{m,n,1}$ is the interpolation super Jack polynomial associated to μ from Remark 4.4.5.

5.2 The CEP for $(\mathfrak{g}, \mathfrak{b}, V)$ with arbitrary $\mathfrak{b}$

In this section, let $\mathfrak{d}$ be an arbitrary Borel sublagebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Let $\underline{\lambda}_{\mathfrak{d}}$ be the $\mathfrak{d}$-highest weight of the irreducible component $\left(V_{m|n}^{\lambda}\right)^* \otimes V_{m|n}^{\lambda}$ appearing in the decomposition of $\mathcal{P}(V)$. The goal of this chapter is to show that we can we express the eigenvalue of $D_{m|n}^{\mu}$ on $\left(V_{m|n}^{\lambda}\right)^* \otimes V_{m|n}^{\lambda}$ as a $SP_{\mu,1}^{*} \circ \tau_{\mathfrak{d}}\left(\underline{\lambda}_{\mathfrak{d}}\right)$.

The first step is to observe the relation between the affine map τ_0 and the Weyl vector of $\mathfrak{gl}(m|n)$. Recall from Example 2.1.21 that the standard Weyl vector ρ_0 of $\mathfrak{gl}(m|n)$ is given by

$$\rho_0 = \sum_{i=1}^{m} \frac{m-n+1-2i}{2}\epsilon_i + \sum_{j=1}^{n} \frac{m+n+1-2j}{2}\delta_j.$$

For simplicity, we denote

$$E_i := \frac{m-n+1-2i}{2} \text{ and } F_j = \frac{m+n+1-2j}{2} \tag{5.2.1}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Remark 5.2.1. It is clear that there is an isomorphism from $\mathfrak{h}_{m|n}^*$ to $\mathbb{C}^{m|n}$ by sending ϵ_i to e_i and δ_j to e_{m+j}. For simplicity, we denote the image of $\alpha \in \mathfrak{h}_{m|n}^*$ under this isomorphism by $\widetilde{\alpha}$. For example $\widetilde{\rho_0} = \sum_{i=1}^{m} E_i e_i + \sum_{j=1}^{n} F_j e_{m+j}$.

By Remark 5.2.1, we can rewrite the affine map τ_0 as follows.

Lemma 5.2.2. *Let $\eta, x \in \mathfrak{h}_{m|n}^*$. The affine map τ_0 defined in Definition 5.1.2 is then*

$$\tau_0(\eta, x) = \widetilde{x} + \widetilde{\rho_0}.$$

Having rephrased the result from [SSS20] in Lemma 5.2.2, we see in a more transparent way that the map τ_0 depends on the choice of Borel subalgebra through the Weyl vector.

Our first main theorem is that an analogous formula holds for any *increasing* Borel subalgebra.

Theorem 5.2.3. *Let $\mathfrak{b}$ be an increasing Borel subalgebra of $\mathfrak{gl}(m|n)$. Let $\rho_{\mathfrak{b}}$ be the associated Weyl vector of $\mathfrak{b}$. Let $\mathfrak{c}$ be another Borel subalgebra of $\mathfrak{gl}(m|n)$ that contains $\mathfrak{h}_{m|n}$. Define the affine map $\tau_{\mathfrak{b}} : \mathfrak{h}^* \to \mathbb{C}^{m|n}$ by*

$$\tau_{\mathfrak{b}}((\eta, x)) := \tilde{x} + \tilde{\rho}_{\mathfrak{b}}$$

for any $\eta, x \in \mathfrak{h}^*_{m|n}$. For any submodule $\left(V^{\lambda}_{m|n}\right)^* \otimes V^{\lambda}_{m|n}$ appearing in $\mathcal{P}(V)$, suppose $\left(-\underline{\lambda}_{m|n}, \underline{\lambda}_{m|n}\right) \in \mathfrak{h}^*$ is the highest weight of $\left(V^{\lambda}_{m|n}\right)^* \otimes V^{\lambda}_{m|n}$ with respect to the Borel subalgebra $\mathfrak{b}_{\mathrm{st}}$. Let $(\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})$ be the highest weight of $\left(V^{\lambda}_{m|n}\right)^* \otimes V^{\lambda}_{m|n}$ with respect to $\mathfrak{c} \oplus \mathfrak{b}$. Then

$$SP^*_{\mu,1} \circ \tau_{\mathfrak{b}}\left((\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})\right) = SP^*_{\mu,1} \circ \tau_0((-\underline{\lambda}_{m|n}, \underline{\lambda}_{m|n})),$$

where $SP^*_{\mu,1}$ is from in Remark 4.4.5.

Consequently, the eigenvalue of Capelli operator $D^{\mu}_{m|n}$ on the irreducible component $\left(V^{\lambda}_{m|n}\right)^* \otimes V^{\lambda}_{m|n}$ with highest weight $(\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})$ is equal to $SP^*_{\mu,1} \circ \tau_{\mathfrak{b}}\left((\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})\right)$.

We first rewrite the monoidal symmetry property for $c_\mu \in \Lambda_{m,n,1}$ to facilitate the proof of our main theorem.

Lemma 5.2.4. *Let $c_\mu(x|y) = c_\mu(x_1, \ldots, x_m|y_1, \ldots, y_n) \in \Lambda_{m,n,1}$. Then c_μ is separately symmetric in $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ respectively, and moreover, we have that*

$$c_\mu(x|y) = c_\mu(x_i - e_i|y_j + e_{m+j}) = c_\mu(x_i + e_i|y_j - e_{m+j})$$

if $x_i + y_j = 0$.

Proof. The separate symmetry follows from the fact that $c_\mu \in \Lambda_{m,n,1}$. Then it suffices to show that $(x|y) \sim (x \mp e_i | y \pm e_{m+j})$ whenever $x_i + y_j = 0$. The result follows from Lemma 4.4.3 by noticing when $\theta = 1$, the equivalence relation $x_i + \theta y_j = \pm\frac{1}{2}(1-\theta)$ is equivalent to $x_i + y_j = 0$. □

Proof of Theorem 5.2.3. Since $SP^*_{\mu,1} \in \Lambda_{m,n,1}$, it suffices to show that

$$\tau_{\mathfrak{b}}((\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})) \sim \tau_0((-\underline{\lambda}_{m|n}, \underline{\lambda}_{m|n})).$$

From Remark 2.1.18, we know that we can obtain any increasing Borel subalgebra from the $\epsilon\delta$-sequence associated to $\mathfrak{b}_{m|n}$ by a sequence of odd reflections, so we may proceed by induction on the number of odd reflections taking $\mathfrak{b}_{m|n}$ to $\mathfrak{b}$. The result holds for $\mathfrak{b} = \mathfrak{b}_{m|n}$. Let $k \in \mathbb{Z}_{\geq 0}$ be arbitrary and $\mathfrak{b}_k$ be an increasing Borel subalgebra of $\mathfrak{gl}(m|n)$. Suppose that the result also holds for $\mathfrak{b}_k$.

Let $\alpha_k = \epsilon_i - \delta_j$ be a simple isotropic root with respect to $\mathfrak{b}_k$ and $\mathfrak{b}_{k+1} = r_\alpha(\mathfrak{b}_k)$. Let ρ_k, ρ_{k+1} be the associated Weyl vector of $\mathfrak{b}_k, \mathfrak{b}_{k+1}$ respectively. Let $\underline{\lambda}_k$ and $\underline{\lambda}_{k+1}$ be the highest weight of $V_{m|n}$ with respect $\mathfrak{b}_k$ and $\mathfrak{b}_{k+1}$ respectively. Thus, it suffices to show that for all $\lambda_k, \lambda_{k+1} \in \mathfrak{h}^*_{m|n}$,

$$\tau_{k+1}\left(\left(\lambda_{k+1}, \underline{\lambda}_{k+1}\right)\right) \sim \tau_k\left((\underline{\lambda}_k, \underline{\lambda}_k)\right).$$

First notice that $\underline{\lambda}_k$ is of the form of $\sum_{i=1}^m p_i\epsilon_i + \sum_{j=1}^n q_j\delta_j$ where $p_i, q_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Therefore, by Corollary 2.2.5, we have that

$$\underline{\lambda}_{k+1} = \begin{cases} \underline{\lambda}_k & \text{if } (\underline{\lambda}_k, \alpha_k) = p_i + q_j = 0; \\ \underline{\lambda}_k - \alpha_k & \text{if } (\underline{\lambda}_k, \alpha_k) = p_i + q_j \neq 0. \end{cases}$$

Suppose that $(\underline{\lambda}_k, \alpha_k) \neq 0$. Then we have $\underline{\lambda}_{k+1} = \underline{\lambda}_k - \alpha_k$ and hence $\widetilde{\underline{\lambda}_k} = \widetilde{\underline{\lambda}_{k+1}} + \widetilde{\rho_{k+1}}$. Thus we have that

$$\begin{aligned} \tau_{k+1}\left((\lambda_{k+1}, \underline{\lambda}_{k+1})\right) &= \widetilde{\underline{\lambda}_{k+1}} + \widetilde{\rho_{k+1}} \\ &= \widetilde{\underline{\lambda}_k} - \widetilde{\alpha_k} + \widetilde{\rho_{k+1}}. \end{aligned}$$

Moreover, by Proposition 2.1.22, we have $\widetilde{\rho_{k+1}} = \widetilde{\rho_k} + \widetilde{\alpha_k}$. Therefore,

$$\begin{aligned} \tau_{k+1}\left((\lambda_{k+1}, \underline{\lambda}_{k+1})\right) &= \widetilde{\underline{\lambda}_k} - \widetilde{\alpha_k} + \widetilde{\rho_k} + \widetilde{\alpha_k} \\ &= \widetilde{\underline{\lambda}_k} + \widetilde{\rho_k} \\ &= \tau_k\left((\lambda_k, \underline{\lambda}_k)\right) \end{aligned}$$

as claimed. Now suppose that $(\underline{\lambda}_k, \alpha_k) = 0$, that is $p_i + q_j = 0$ and $\underline{\lambda}_{k+1} = \underline{\lambda}_k$. Then we have

$$\begin{aligned} \tau_{k+1}\left((\lambda_{k+1}, \underline{\lambda}_{k+1})\right) = \widetilde{\underline{\lambda}_{k+1}} + \widetilde{\rho_{k+1}} &= \widetilde{\underline{\lambda}_k} + \widetilde{\rho_{k+1}} \\ &= \widetilde{\underline{\lambda}_k} + \widetilde{\rho_k} + \widetilde{\alpha_k} \\ &= \tau_k\left((\lambda_k, \underline{\lambda}_k)\right) + e_i - e_{m+j}, \end{aligned} \tag{5.2.2}$$

which implies that $\tau_{k+1}\left((\underline{\lambda}_{k+1}, \underline{\lambda}_{k+1})\right)$ and $\tau_k\left((\underline{\lambda}_k, \underline{\lambda}_k)\right)$ only differ in their e_i and e_{m+j} coefficients. We then analyze these coefficients. By definition, $\tau_k\left((\lambda_k, \underline{\lambda}_k)\right) = \widetilde{\underline{\lambda}_k} + \widetilde{\rho_k}$.

Recall from Lemma 2.1.23, the e_i and e_{m+j} coefficients of $\widetilde{\rho_k}$ are $E_i + (j-1)$ and $F_j - (m-i)$ respectively. Therefore, by the fact that $\widetilde{\underline{\lambda}_k} = \sum_{i=1}^m p_i e_i + \sum_{j=1}^n q_j e_{m+j}$, the e_i and e_{m+j} coefficients of $\tau_k\left((y_k, \underline{\lambda}_k)\right)$ are precisely

$$p_i + E_i + (j-1) \qquad \text{and} \qquad q_j + F_j - (m-i)$$

respectively. For convenience, let $A_i = p_i + E_i + (j-1)$ and $B_j = q_j + F_j - (m-i)$. Thus, by the fact that

$$E_i + F_j = m + 1 - i - j$$

for all $1 \leq i \leq m, 1 \leq j \leq n$ and the fact $p_i + q_j = 0$, we have that

$$A_i + B_j = p_i + q_j + E_i + F_j - 1 - m + i + j = 0.$$

Therefore, by Corollary 5.2.4 we have that

$$\tau_k\left((\lambda_k, \underline{\lambda}_k)\right) \sim \tau_k\left((\lambda_k, \underline{\lambda}_k)\right) + e_i - e_{m+j} = \tau_{k+1}\left((\lambda_{k+1}, \underline{\lambda}_{k+1})\right)$$

which completes the proof. $\square$

In order to fully complete the argument that the refined solution to the CEP for $(\mathfrak{g}, \mathfrak{b}, V)$ can be found for *any* Borel subalgebras, we consider different Borel subalgebras in the same conjugacy class under the Weyl group. That is, we include those that correspond to $\epsilon\delta$ sequences that are *not* necessarily increasing.

Corollary 5.2.5. *Retain the setup in Theorem 5.2.3. Let $\mathfrak{c}'$ be any Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}_{m|n}$. Let $\mathfrak{b}'$ be any Borel subalgebra which is conjugate to $\mathfrak{b}$ under the Weyl group action. Let ρ, ρ' be the Weyl vectors associated to $\mathfrak{b}, \mathfrak{b}'$ respectively. Let $(\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})$*

and $(\underline{\lambda}_{\mathfrak{c}'}, \underline{\lambda}_{\mathfrak{b}'})$ *be the highest weights of finite-dimensional irreducible* $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n)$*-modules* $\left(V^{\lambda}_{m|n}\right)^* \otimes V^{\lambda}_{m|n}$ *with respect to* $\mathfrak{c} \oplus \mathfrak{b}$ *and* $\mathfrak{c}' \oplus \mathfrak{b}'$ *respectively. For all* $\eta, x \in \mathfrak{h}^*_{m|n}$, *define*

$$\tau_{\mathfrak{b}'}\left((\eta, x)\right) = \widetilde{x} + \widetilde{\rho'}.$$

Then

$$SP^*_{\mu,1} \circ \tau_{\mathfrak{b}}\left((\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})\right) = SP^*_{\mu,1} \circ \tau_{\mathfrak{b}'}\left((\underline{\lambda}_{\mathfrak{c}'}, \underline{\lambda}_{\mathfrak{b}'})\right).$$

Consequently, the eigenvalue of Capelli operator $D^{\mu}_{m|n}$ *on the irreducible component* $\left(V^{\lambda}_{m|n}\right)^* \otimes V_{m|n}$ *with highest weight* $(\underline{\lambda}_{\mathfrak{c}'}, \underline{\lambda}_{\mathfrak{b}'})$ *is equal to* $SP^*_{\mu,1} \circ \tau_{\mathfrak{b}}\left((\underline{\lambda}_{\mathfrak{c}}, \underline{\lambda}_{\mathfrak{b}})\right)$.

Proof. Since $\mathfrak{b}'$ is conjugate to $\mathfrak{b}$, there exists $\sigma \in S_m$ and $\sigma' \in S_n$ such that $\underline{\lambda}_{\mathfrak{b}'}$ is obtained from $\underline{\lambda}_{\mathfrak{b}}$ by permuting $\epsilon_1, \ldots, \epsilon_m$ via σ, and permuting $\delta_1, \ldots, \delta_n$ via σ'. Similarly, ρ' is obtained from ρ in the same way. Therefore we have

$$\tau_{\mathfrak{b}'}\left((\lambda', \underline{\lambda}_{\mathfrak{b}'})\right) = \widetilde{\underline{\lambda}_{\mathfrak{b}'}} + \widetilde{\rho'} \sim \widetilde{\underline{\lambda}_{\mathfrak{b}}} + \widetilde{\rho} = \tau_{\mathfrak{b}}\left((\lambda, \underline{\lambda}_{\mathfrak{b}})\right)$$

where the equivalent relation $\sim$ is used by the fact that the interpolation super Jack polynomial $SP^*_{\mu,1} \in \Lambda_{m,n,1}$ is separately symmetric. □

In Theorem 5.2.3 and Corollary 5.2.5, we showed that for any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, we can find an affine map $\tau_{\mathfrak{b}}$ such that $SP^*_{\mu,1} \circ \tau_{\mathfrak{b}}\left((y, \lambda_{\mathfrak{b}})\right) = SP^*_{\mu,1} \circ \tau_0\left(\left(-\lambda_{m|n}, \lambda_{m|n}\right)\right)$ which refines the result from [SSS20]. However, in the case of $\mathfrak{g} = \mathfrak{gl}(m|2n)$, the situation will be much more complicated. The next chapter will focus on investigating the solution to the CEP in the case of $\mathfrak{g} = \mathfrak{gl}(m|2n)$.

Chapter 6

The CEP for $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$

In this chapter, let $\mathfrak{g} = \mathfrak{gl}(m|2n)$. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}_{\text{op}}$ be the opposite standard (i.e., lower triangular) Borel subalgebra of $\mathfrak{g}$. Let $V := \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)$. We follow a similar structure of Chapter 5 to first give the solution to the CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{op}}, V)$, and then move on to study a refined solution to the CEP for $(\mathfrak{g}, V)$ for different Borel subalgebras of $\mathfrak{g}$.

6.1 The Solution to the CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{op}}, V)$

To give the solution to the CEP for $(\mathfrak{g}, \mathfrak{b}_{\text{op}}, V)$, we first show that $\mathcal{P}(V)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module.

Theorem 6.1.1. *The polynomial algebra $\mathcal{P}\left(\mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$ is a completely reducible and multiplicity-free $\mathfrak{g}$-module, with the decomposition*

$$\mathcal{P}\left(\mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right) = \bigoplus_{\lambda} \left(V_{m|2n}^{\lambda}\right)^*, \tag{6.1.1}$$

where λ runs over $\mathcal{H}(m|2n)$ and has all even parts. If we write λ in the form

$(2\lambda_1, \ldots, 2\lambda_{m+2n})$, *then the highest weight* $\underline{\lambda}_0$ *of* $\left(V^{\lambda}_{m|2n}\right)^*$ *with respect to* $\mathfrak{b}_{\text{op}}$ *is*

$$\underline{\lambda}_0 = -\sum_{i=1}^{m} 2\lambda_i \epsilon_i - \sum_{j=1}^{n} \mu_j \left(\delta_{2j-1} + \delta_{2j}\right) \tag{6.1.2}$$

where $\mu_j = \langle \lambda'_j - m \rangle \coloneqq \max\{0, \lambda'_j - m\}$.

Proof. By [CW01, Theorem 3.4], the symmetric algebra of the symmetric square of the natural representation $\mathbb{C}^{m|2n}$ of the Lie superalgebra $\mathfrak{gl}(m|2n)$ is a completely reducible multiplicity-free $\mathfrak{gl}(m|2n)$-module, whose decomposition is

$$\mathcal{S}\left(\mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right) = \bigoplus_{\lambda} V^{\lambda}_{m|2n}$$

where λ runs over $\mathcal{H}(m|2n)$ and has all even parts. Write $\lambda = (2\lambda_1, \ldots, 2\lambda_{m+2n})$. Then the lowest weight of $V^{\lambda}_{m|n}$ with respect to $\mathfrak{b}_{\text{op}}$ is given by

$$\sum_{i=1}^{m} 2\lambda_i \epsilon_i + \sum_{j=1}^{n} \langle \lambda'_j - m \rangle \left(\delta_{2j-1} + \delta_{2j}\right).$$

Then the result follows from the fact that for any $\mathfrak{g}$-module V, $\mathcal{P}(V^*) \cong \mathcal{S}(V)$ as $\mathfrak{g}$-modules, and the fact that the highest weight of $\left(V^{\lambda}_{m|2n}\right)^*$ with respect to $\mathfrak{b}_{\text{op}}$ is negative of the lowest weight of $V^{\lambda}_{m|2n}$ with respect to $\mathfrak{b}_{\text{op}}$ for any finite dimensional irreducible $\mathfrak{g}$-module $V^{\lambda}_{m|2n}$. □

Let $\Omega_{m|2n}$ be the set of all $\mathfrak{b}_{\text{op}}$-highest weights $\underline{\lambda}_0$ appearing in the decomposition (6.1.1). The span of $\Omega_{m|2n}$ is a subspace of $\mathfrak{h}^*$ denoted $\mathfrak{a}^*$.

Definition 6.1.2. *For* $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{C}$, *let* $\underline{\lambda}$ *be in* $\mathfrak{a}^*$ *of the form* $\underline{\lambda} =$

$\sum_{i=1}^{m} a_i\epsilon_i + \sum_{j=1}^{n} b_j(\delta_{2-1} + \delta_{2j})$. *We define an affine map* τ_0 *from* $\mathfrak{a}^*$ *to* $\mathbb{C}^{m|n}$ *by*

$$\tau_0(\underline{\lambda}) = -\sum_{i=1}^{m}\left(a_i - \frac{m+1-2n-2i}{4}e_i\right) - \sum_{j=1}^{n}\left(b_j - \frac{m+2+2n-4j}{2}\right)e_{m+j}, \tag{6.1.3}$$

where the set $\{e_i, e_{m+j}\}_{1\leq i\leq m, 1\leq j\leq n}$ *is the standard basis for* $\mathbb{C}^{m|n}$.

For simplicity, for all $1 \leq i \leq m$ and $1 \leq j \leq n$, let

$$E_i \coloneqq \frac{m+1-2n-2i}{4} \text{ and } F_j \coloneqq \frac{m+2+2n-4j}{2} \tag{6.1.4}$$

be the partial terms appeared in Equation (6.1.3) respectively. Notice that

$$E_i + \frac{1}{2}F_j = \frac{1}{2}(m-i) - j + \frac{3}{4}. \tag{6.1.5}$$

Theorem 6.1.3. *[SSS20, Theorem 1.13.]Let* $\mathfrak{g} = \mathfrak{gl}(m|2n)$. *Let* $\mathfrak{b}_{\text{op}}$ *be the opposite standard Borel subalgebra and* $V \coloneqq \mathcal{S}^2\left(\mathbb{C}^{m|n}\right)$. *Then for each* $\lambda, \mu \in \mathcal{H}(m|2n)$, *the eigenvalue of the Capelli operator* $D^{\mu}_{m|2n}$ *(Definition 3.4.2) on* $\left(V^{\lambda}_{m|2n}\right)^*$ *with highest weight* $\underline{\lambda}_0$ *with respect to* $\mathfrak{b}_{\text{op}}$ *is equal to*

$$SP^*_{\mu,\frac{1}{2}}(\tau_0(\underline{\lambda}_0)),$$

where $SP^*_{\mu,\frac{1}{2}} \in \Lambda_{m,n,\frac{1}{2}}$ *is the interpolation super Jack polynomial associated to* μ *from Remark 4.4.5.*

We finish this section by rewriting the monoidal symmetry property for $c_\mu \in \Lambda_{m,n,\frac{1}{2}}$ to facilitate the proof of our main theorem in this chapter.

Lemma 6.1.4. *Let* $c_\mu(x|y) = c_\mu(x_1, \ldots, x_m|y_1, \ldots, y_n) \in \Lambda_{m,n,\frac{1}{2}}$. *Then* c_μ *is separately* *symmetric in* $\{x_1, \ldots, x_m\}$ *and* $\{y_1, \ldots, y_n\}$ *respectively, and moreover, we have that*

$$c_\mu(x|y) = \begin{cases} c_\mu(x - e_i|y + e_{m+j}) & \text{if } x_i + \frac{1}{2}y_j = \frac{1}{4} \\ c_\mu(x + e_i|y - e_{m+j}) & \text{if } x_i + \frac{1}{2}y_j = -\frac{1}{4}. \end{cases}$$

Proof. The separate symmetry follows from the fact that $c_\mu \in \Lambda_{m,n,\frac{1}{2}}$. Then it suffices to show that $(x|y) \sim (x \mp e_i|y \pm e_{m+j})$ whenever $x_i + \frac{1}{2}y_j = \pm\frac{1}{4}$. Then the result follows from Lemma 4.4.3 by noticing when $\theta = \frac{1}{2}$, the equality $x_i + \theta y_j = \pm\frac{1}{2}(1-\theta)$ is equivalent to $x_i + \frac{1}{2}y_j = \pm\frac{1}{4}$. □

6.2 Explicit form of $\tau_0(\underline{\lambda}_0)$

In this section, we give the explicit matrix form of every affine map from $\mathfrak{h}^*$ to $\mathbb{C}^{m|n}$ that extends the map τ_0 given in Definition 6.1.2. Namely, we give the set of matrices M and vector a X_0 such that $\tau_0(\underline{\lambda}_0) = M\underline{\lambda}_0 + X_0$ for all $\underline{\lambda}_0 \in \mathfrak{a}^*$.

Definition 6.2.1. *Let* $\mathcal{C}$ *be the set of all matrices of the form*

$$M = \begin{bmatrix} -\frac{1}{2}I_m & 0_{m\times 2n} \\ 0_{n\times m} & D \end{bmatrix} + \begin{bmatrix} 0_{(m+n)\times m} & A \end{bmatrix}$$

where $D = (d_{ij})$ *is the* $n \times 2n$ *matrix whose entries satisfy*

$$d_{ij} = \begin{cases} -\frac{1}{2} & \text{if } j = 2i-1 \text{ or } 2i \\ 0 & \text{otherwise,} \end{cases} \tag{6.2.1}$$

and A is any $(m+n)\times 2n$ matrix whose columns A_i satisfy the relation

$$A_{2i} = -A_{2i-1} \tag{6.2.2}$$

for all $1 \leq i \leq n$.

Recall that

$$\tau_0(\underline{\lambda}_0) = \sum_{i=1}^{m}(\lambda_i + E_i)e_i + \sum_{j=1}^{n}(\mu_j + F_j)e_{m+j},$$

where $\underline{\lambda}_0$ is of the form given in Equation (6.1.2). Set $X_0 = \tau_0(0)$.

Lemma 6.2.2. *The set $\mathcal{C}$ is exactly the set of all matrices M such that $\tau_0(\underline{\lambda}_0) = M\underline{\lambda}_0 + X_0$ for all highest weights $\underline{\lambda}_0 \in \Omega_{m|2n}$.*

Proof. We see that τ_0 is an affine transformation from $\mathfrak{h}^*$ onto $\mathbb{C}^{m|n}$. If we write this as $\tau_0(\underline{\lambda}_0) = M_0\underline{\lambda}_0 + X_0$, then $X_0 = (E_1, \cdots, E_m, F_1, \cdots, F_n)$ is uniquely determined but M_0 is not. Namely, if K is any matrix such that $K\underline{\lambda}_0 = 0$ for all $\underline{\lambda}_0 \in \Omega_{m|2n}$, then $\tau_0(\underline{\lambda}_0) = (M_0 + K)\underline{\lambda}_0 + X_0$ as well, and by linearity these are all possible choices. Since $\mathfrak{a}^* = \text{span}\{\epsilon_1, \cdots, \epsilon_m, \delta_1 + \delta_2, \cdots, \delta_{2n-1} + \delta_{2n}\}$, we deduce that the set $\mathcal{K}$ of all matrices vanishing on $\mathfrak{a}^*$ is

$$\mathcal{K} = \left\{ \begin{bmatrix} 0_{(n+m)\times m} & A \end{bmatrix} \mid A_{2k-1} = -A_{2k} \right.$$
$$\left. \text{for all } 1 \leq k \leq n, \text{ where } A_i \text{ is the } i\text{th column of } A \right\}.$$

We make the choice

$$\mathcal{M} = \begin{bmatrix} -\frac{1}{2}I_m & 0_{m\times 2n} \\ 0_{n\times m} & D \end{bmatrix}, \tag{6.2.3}$$

where $D = (d_{ij})$ is the $n \times 2n$ matrix whose entries satisfy $d_{ij} = -\frac{1}{2}$ if $j = 2i-1$ and

$2i$, or 0 otherwise, and define $\mathcal{C} = \{M_0 + K \mid K \in \mathcal{K}\}$. Then any $M \in \mathcal{C}$ satisfies $\tau_0(\lambda_0) = M\lambda_0 + X_0$. □

Remark 6.2.3. Let $\mathcal{M}$ be the particular matrix in Equation 6.2.3. Then a direct calculation shows that $\mathcal{M}\rho_{\text{op}} = X_0$, where ρ_{op} is the Weyl vector of $\mathfrak{gl}(m|2n)$ with respect to $\mathfrak{b}_{\text{op}}$.

However, this is not true of a general element $M_0 \in \mathcal{C}$, since ρ_{op} is not an element of $\mathfrak{a}^*$. In the following sections, we will see that the single matrix $\mathcal{M}$ is not suitable for other Borel subalgebras, but that other subsets of $\mathcal{C}$ will be.

In the following, we will indiscriminately use M_0 to refer to any element of $\mathcal{C}$. The rest of this chapter will focus on finding affine maps with respect to different Borel subalgebras other than $\mathfrak{b}_{\text{op}}$. In next section, we first give an explicit example of $\mathfrak{gl}(1|2)$. We also include an explicit calculation for $\mathfrak{gl}(1|2n)$ in Appendix A for the readers to understand the idea we used in later sections to prove the general case for $\mathfrak{gl}(m|2n)$.

6.3 The case $\mathfrak{gl}(1|2)$

In this section, let $\mathfrak{g} = \mathfrak{gl}(1|2)$. We give a complete calculation of the affine maps. This may give us an idea of how we set $X_\mathfrak{b}$ for a given $\mathfrak{b}$. Beginning with the Borel subalgebra $\mathfrak{b}_{\text{op}} = \delta_2\delta_1\epsilon_1$, the highest weight of $\left(V_{1|2}^{\lambda}\right)^*$ with respect to $\mathfrak{b}_{\text{op}}$ is

$$\underset{\sim}{\lambda}_0 = \left(-2\lambda_1, \mid -\mu_1, -\mu_1\right),$$

where $\lambda_1, \mu_1 \geq 0$ and if $\lambda_1 = 0$, so is μ_1. First recall from Equation (6.1.4) that we have $E_1 = -\frac{1}{2}$ and $F_1 = \frac{1}{2}$, so that

$$\tau_0(\underline{\lambda_0}) = (\lambda_1 + E_1, \mu_1 + F_1).$$

Let $M_0 = \begin{pmatrix} -\frac{1}{2} & b & -b \\ 0 & e & -1-e \end{pmatrix}$ be a matrix in $\mathcal{C}$ such that b, e are arbitrary. Let $X_0 = \tau_0(0) = \begin{pmatrix} E_1 \\ F_1 \end{pmatrix}$. Then we have shown that

$$\tau_0(\underline{\lambda_0}) = \begin{pmatrix} -\frac{1}{2} & b & -b \\ 0 & e & -1-e \end{pmatrix} \begin{pmatrix} -2\lambda_1 \\ -\mu_1 \\ -\mu_1 \end{pmatrix} + \begin{pmatrix} E_1 \\ F_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + E_1 \\ \mu_1 + F_1 \end{pmatrix}.$$

6.3.1 The Borel subalgebra $\delta_2\epsilon_1\delta_1$

Consider $\mathfrak{b}_1 = r_{\delta_1-\epsilon_1}(\mathfrak{b}_{\text{op}}) = \delta_2\epsilon_1\delta_1$. By Corollary 2.2.5, The highest weight $\underline{\lambda_1}$ of $\left(V^{\lambda}_{1|2}\right)^*$ with respect to $\mathfrak{b}_1$ is given by

$$\underline{\lambda_1} = \begin{cases} \underline{\lambda_0} & \text{if } (\underline{\lambda_0}, \delta_1 - \epsilon_1) = 0 \\ \underline{\lambda_0} - (\delta_1 - \epsilon_1) & \text{if } (\underline{\lambda_0}, \delta_1 - \epsilon_1) \neq 0 \end{cases} \tag{6.3.1}$$

More precisely, the condition for $\underline{\lambda_1} = \underline{\lambda_0}$ is $(\underline{\lambda_0}, \delta_1 - \epsilon_1) = 2\lambda_1 + \mu_1 = 0$. Thus, by the fact that $\lambda_1, \mu_1 \in \mathbb{Z}_{\geq 0}$, we conclude that $\underline{\lambda_1}$ takes the form of $\underline{\lambda_1} = (0 \mid 0, 0)$ if $\underline{\lambda_0} = (0 \mid 0, 0)$, and

$$\underline{\lambda_1} = (-2\lambda_1 + 1 \mid -\mu_1 - 1, -\mu_1) \tag{6.3.2}$$

otherwise. Call $\underline{\lambda}_0$ generic for $\mathfrak{b}_1$ in the second case of Equation (6.3.1), and nongeneric otherwise. To find an affine function τ_1 that on input of $\underline{\lambda}_1$ gives an output that is monoidally equivalent to $\tau_0(\underline{\lambda}_0)$, let

$$M_1 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \text{ and } X_1 = \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

be arbitrary. Then for $\underline{\lambda}_1 = (-2\lambda_1 + 1 \mid -\mu_1 - 1, -\mu_1)$, we want

$$M\underline{\lambda}_1 + X_1 \sim \begin{pmatrix} \lambda_1 + E_1 \\ \mu_1 + F_1 \end{pmatrix}$$

for all generic highest weights $\underline{\lambda}_0$. By Lemma 6.1.4, we verify that

$$(\lambda_1 + E_1) + \frac{1}{2}(\mu_1 + F_1) = \lambda_1 + \frac{1}{2}\mu_1 - \frac{1}{4} > \frac{1}{4} \text{ if and only if } \lambda_1 > \frac{1}{2} - \frac{1}{2}\mu_1,$$

and

$$(\lambda_1 + E_1) + \frac{1}{2}(\mu_1 + F_1) = \lambda_1 + \frac{1}{2}\mu_1 - \frac{1}{4} > -\frac{1}{4} \text{ if and only if } \lambda_1 > -\frac{1}{2}\mu_1.$$

Thus, for generic highest weights $\underline{\lambda}_0$ such that $\lambda_1 > \frac{1}{2} - \frac{1}{2}\mu_1$, no monoidal symmetry can be applied. Therefore, we must in fact have equality

$$\begin{pmatrix} -2a\lambda_1 + a - (b+c)\mu_1 - b + A_1 \\ -2d\lambda_1 + d - (e+f)\mu_1 - e + B_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + E_1 \\ \mu_1 + F_1. \end{pmatrix} \tag{6.3.3}$$

This equality holds for infinitely many linearly independent pairs (λ_1, μ_1), so we deduce that

$$a = -\frac{1}{2}, c = -b, d = 0, f = -1 - e.$$

Therefore, $M_1 = \begin{pmatrix} -\frac{1}{2} & b & -b \\ 0 & e & -1-e \end{pmatrix}$ is once again an arbitrary element of $\mathcal{C}$, and the vector X_1 is

$$X_1 = \begin{pmatrix} E_1 + \frac{1}{2} + b \\ F_1 + e \end{pmatrix}.$$

Now let $\underline{\lambda}_1 = \underline{\lambda}_0 = (0 \mid 0, 0)$ be the only nongeneric highest weight. Then we must have $\tau_1(\underline{\lambda}_1) \sim \tau_0((0 \mid 0, 0))$, which implies that

$$\begin{pmatrix} E_1 + \frac{1}{2} + b \\ F_1 + e \end{pmatrix} \sim \begin{pmatrix} E_1 \\ F_1 \end{pmatrix}. \tag{6.3.4}$$

We calculate $E_1 + \frac{1}{2}F_1 = -\frac{1}{4}$ and thus

$$\begin{pmatrix} E_1 \\ F_1 \end{pmatrix} \sim \begin{pmatrix} E_1 + 1 \\ F_1 - 1 \end{pmatrix}.$$

In fact, these are the only two elements in the equivalence class. Therefore, we have two solutions:

$$M_1 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \end{pmatrix} \text{ and } X_1 = \begin{pmatrix} E_1 + 1 \\ F_1 - 1 \end{pmatrix} \tag{6.3.5}$$

or

$$M_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix} \text{ and } X_1 = \begin{pmatrix} E_1 \\ F_1 \end{pmatrix}. \tag{6.3.6}$$

Thus we have that $SP^*_{\mu,\frac{1}{2}} \circ \tau_1(\underline{\lambda}_1) = SP^*_{\mu,\frac{1}{2}} \circ \tau_0(\underline{\lambda}_0)$ for either of the pairs (M_1, X_1) above.

6.3.2 The Borel subalgebra $\mathfrak{b}_2 = \epsilon_1\delta_2\delta_1$

Consider $\mathfrak{b}_2 = r_{\delta_2-\epsilon_1}(\mathfrak{b}_1) = \epsilon_1\delta_2\delta_1$. We follow a similar argument as in Section 6.3.1. We calculate the highest weights $\underline{\lambda}_{\mathfrak{b}_2}$ using Corollary 2.2.5. When $\underline{\lambda}_1$ is generic (Equation (6.3.2)), the coefficient of ϵ_1 is odd (and so nonzero). Therefore $(\underline{\lambda}_1, \delta_2 - \epsilon_1) \neq 0$, so every generic highest weight with respect to $\mathfrak{b}_1$ is also generic with respect to $\mathfrak{b}_2$. The $(0 \mid 0, 0)$ weight remains the only nongeneric weight. Thus:

$$\underline{\lambda}_2 = \begin{cases} (0 \mid 0, 0) & \text{if } \underline{\lambda}_0 = 0 \\ (-2\lambda_1 + 2, \mid -\mu_1 - 1, -\mu_1 - 1) & \text{otherwise.} \end{cases}$$

To find an affine function τ_2 that on input of $\underline{\lambda}_2$ gives an output that is monoidally equivalent to $\tau_0(\underline{\lambda}_0)$, let $M_2 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $X_2 = \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$. As before, we conclude that for the infinitely many generic highest weights we must have equality $M_2\lambda_2 + X_2 = M_0\lambda_0 + X_0$. A calculation as before shows that $M_2 \in \mathcal{C}$, so our desired equality simplifies to

$$\begin{pmatrix} -\frac{1}{2} & b & -b \\ 0 & e & -1-e \end{pmatrix} \begin{pmatrix} -2\lambda_1 + 2 \\ -\mu_1 - 1 \\ -\mu_1 - 1 \end{pmatrix} + \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + E_1 \\ \mu_1 + F_1 \end{pmatrix}$$

for some b, e, which implies $X_2 = \begin{pmatrix} E_1 + 1 \\ F_1 - 1 \end{pmatrix}$. For the nongeneric case, $M_2\,(0 \mid 0, 0) + X_2 = X_2$ which we have already seen is monoidally equivalent to $X_0 = \tau_0((0 \mid 0, 0))$.

$\mathfrak{b}$	M	X
$\mathfrak{b}_{\mathrm{op}} = \delta_2\delta_1\epsilon_1$	$\begin{pmatrix} -\frac{1}{2} & b & -b \\ 0 & e & -1-e \end{pmatrix}$	$\begin{pmatrix} E_1 \\ F_1 \end{pmatrix}$
$\mathfrak{b}_1 = \delta_2\epsilon_1\delta_1$	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} E_1 \\ F_1 \end{pmatrix}$
$\mathfrak{b}_1 = \delta_2\epsilon_1\delta_1$	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} E_1+1 \\ F_1-1 \end{pmatrix}$
$\mathfrak{b}_2 = \epsilon_1\delta_2\delta_1$	$\begin{pmatrix} -\frac{1}{2} & b & -b \\ 0 & e & -1-e \end{pmatrix}$	$\begin{pmatrix} E_1+1 \\ F_1-1 \end{pmatrix}$

Table 6.1: Summary of the affine maps for $\mathfrak{gl}(1|2)$

Thus we have that $SP^*_{\mu,\frac{1}{2}} \circ \tau_2(\underline{\lambda}_2) = SP^*_{\mu,\frac{1}{2}} \circ \tau_0(\underline{\lambda}_0)$ for M_2, X_2 as above.

We summarize these results in Table 6.1.

6.4 A formula for the $\mathfrak{b}$-highest weight

In this chapter, we first define *decreasing* Borel subalgebras $\mathfrak{b}$ and what it means for a $\mathfrak{b}_{\mathrm{op}}$-highest weight to be generic with $\mathfrak{b}$. We then give a formula for highest weights of $\left(V^{\lambda}_{m|2n}\right)^*$ with respect to decreasing Borel subalgebras.

6.4.1 Borel subalgebras and decreasing $\delta\epsilon$ sequences

In this subsection, we define decreasing Borel subalgebras and give a method to obtain a Borel subalgeba corresponding to a given decreasing sequence as a result of a sequence of simple odd reflections.

Definition 6.4.1. *A decreasing $\delta\epsilon$ sequence for $\mathfrak{gl}(m|2n)$ is a permutation of the set $\{\epsilon_i, \delta_k \mid 1 \leq i \leq m, 1 \leq k \leq 2n\}$ satisfying the following conditions:*

- *if $k < k'$ then $\delta_{k'}$ precedes δ_k in the sequence (that is, the indices of δ terms is decreasing);*

- *if $i < i'$ then $\epsilon_{i'}$ precedes ϵ_i in the sequence (that is, the indices of the ϵ terms is decreasing).*

We call a Borel subalgebra $\mathfrak{b}$ a *decreasing Borel subalgebra* if the associated $\delta\epsilon$-sequence of $\mathfrak{b}$ is decreasing.

Example 6.4.2. Let $\mathfrak{b}_{\mathrm{op}}$ be the opposite standard Borel subalgebra. The $\delta\epsilon$ sequence corresponding to $\mathfrak{b}_{\mathrm{op}}$ is

$$\delta_{2n}\delta_{2n-1}\cdots\delta_1\epsilon_m\cdots\epsilon_1.$$

Notice that this is a decreasing $\delta\epsilon$-sequence. ♠

By 2.1.17, conjugacy classes of Borel subalgebras of $\mathfrak{gl}(m|2n)$ under the action of the Weyl group are in one-to-one correspondence with decreasing $\delta\epsilon$ sequences. The initial Borel subalgebra $\mathfrak{b}_{\mathrm{op}}$ corresponds to the initial $\delta\epsilon$ sequence, and the transpositions that take one decreasing $\delta\epsilon$-sequence to another one correspond to odd reflections taking one Borel subalgebra to another. Moreover, let $\mathcal{B}$ be the set of all decreasing Borel subalgebras of $\mathfrak{gl}(m|2n)$.

The following lemmas give also the definition of key values $\ell_{i,\mathfrak{b}}$, $j_{k,\mathfrak{b}}$ and $r_{\mathfrak{b}}$ that will be needed in the sequel.

Definition 6.4.3. *Let $\mathfrak{b} \in \mathcal{B}$ be a decreasing Borel subalgebra of $\mathfrak{gl}(m|2n)$.*

1. *For each $1 \leq i \leq m$, let $\ell_{i,\mathfrak{b}}$ be the number of δs to the right of ϵ_i.*

2. *For each $1 \leq k \leq 2n$, let $j_{k,\mathfrak{b}}$ be the number of ϵs to the left of δ_k.*

When $\mathfrak{b}$ is clear from the context, we may write ℓ_i and j_k for $\ell_{i,\mathfrak{b}}$ and $j_{k,\mathfrak{b}}$ respectively.

Lemma 6.4.4. *Let $\mathfrak{b} \in \mathcal{B}$ be a decreasing Borel subalgebra of $\mathfrak{gl}(m|2n)$ with decreasing $\delta\epsilon$ sequence S. Then $\mathfrak{b}$ and S are completely determined by specifying $(\ell_1, \cdots, \ell_m)$ (or equivalently $(j_1, \cdots, j_{2n})$).*

Proof. Note that we can equivalently say that

$$j_k = \#\{i \mid \ell_i \geq k\}$$

since ϵ_i is left of δ_k when δ_k (and hence, $\delta_1, \cdots, \delta_k$) is to the right of ϵ_i. Thus the sequence of values of j is determined by the sequence of values of ℓ. □

Next we give a list of roots which plays a vital role in finding highest weights with respect to different Borel subalgebras.

Definition 6.4.5. *Let $\mathfrak{b} \in \mathcal{B}$ be a decreasing Borel subalgebra of $\mathfrak{gl}(m|2n)$. Let $\mathcal{G}_\mathfrak{b}$ be the set of the following roots*

$$\begin{aligned}
&\delta_1 - \epsilon_m, \delta_2 - \epsilon_m, \cdots, \delta_{\ell_m} - \epsilon_m; \qquad (*)\\
&\delta_1 - \epsilon_{m-1}, \delta_2 - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1}} - \epsilon_{m-1};\\
&\cdots\\
&\delta_1 - \epsilon_1, \delta_2 - \epsilon_1, \cdots, \delta_{\ell_1} - \epsilon_1.
\end{aligned}$$

Define $r_\mathfrak{b} := \sum_{\alpha \in \mathcal{G}_\mathfrak{b}} \alpha$.

By direct computation, we have the following corollary.

Corollary 6.4.6. *Let $\mathcal{G}_\mathfrak{b}$ be defined as in Definition 6.4.5. Then*

$$r_\mathfrak{b} = \sum_{\alpha \in \mathcal{G}_\mathfrak{b}} \alpha = -\sum_{i=1}^{m} \ell_{i,\mathfrak{b}} \epsilon_i + \sum_{k=1}^{2n} j_{k,\mathfrak{b}} \delta_k.$$

Example 6.4.7. Let $\mathfrak{b} = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\epsilon_4\epsilon_3\delta_4\delta_3\delta_2\delta_1\epsilon_2\epsilon_1$. Then (*) is

$$\delta_1 - \epsilon_6, \delta_2 - \epsilon_6, \cdots, \delta_8 - \epsilon_6$$
$$\delta_1 - \epsilon_5, \delta_2 - \epsilon_5, \cdots, \delta_8 - \epsilon_5$$
$$\delta_1 - \epsilon_4, \cdots, \delta_4 - \epsilon_4$$
$$\delta_1 - \epsilon_3, \cdots, \delta_4 - \epsilon_3.$$

Moreover, in the notation of Corollary 6.4.6 we have that

$$(\ell_{1,\mathfrak{b}}, \ell_{2,\mathfrak{b}}, \ell_{3,\mathfrak{b}}, \ell_{4,\mathfrak{b}}, \ell_{5,\mathfrak{b}}, \ell_{6,\mathfrak{b}}) = (0, 0, 4, 4, 8, 8)$$

and

$$(j_{1,\mathfrak{b}}, j_{2,\mathfrak{b}}, j_{3,\mathfrak{b}}, j_{4,\mathfrak{b}}, j_{5,\mathfrak{b}}, j_{6,\mathfrak{b}}, j_{7,\mathfrak{b}}, j_{8,\mathfrak{b}}) = (4, 4, 4, 4, 2, 2, 2, 2)\,.$$

Thus we have

$$r_{\mathfrak{b}} = \sum_{i=1}^{4} 4\delta_i + \sum_{i=5}^{8} 2\delta_i - 8\epsilon_6 - 8\epsilon_5 - 4\epsilon_4 - 4\epsilon_3. \qquad \spadesuit$$

Lemma 6.4.8. *Let $\mathfrak{b} \in \mathcal{B}$ be a decreasing Borel subalgebra of $\mathfrak{gl}(m|2n)$. Let $\mathfrak{b}_{\text{op}}$ be the opposite standard Borel subalgebra. Let $\mathfrak{b} = r_{\alpha_t} r_{\alpha_{t-1}} \cdots r_{\alpha_1}(\mathfrak{b}_{\text{op}})$ where α_i's are simple isotropic roots and take the from of $\delta_j - \epsilon_k$ for some $1 \leq j \leq 2n$ and $1 \leq k \leq m$. We have that $\mathcal{G}_{\mathfrak{b}} = \{\alpha_1, \ldots, \alpha_t\}$.*

Proof. The result follows from the fact that applying odd reflection is equivalent to swapping the position of δ and ϵ, and Lemma 6.4.4 that any Borel subalgebra $\mathfrak{b}$ is completely determined by specifying $(\ell_1, \cdots, \ell_m)$ (or equivalently $(j_1, \cdots, j_{2n})$). □

Definition 6.4.9. *A highest weight $\underline{\lambda}_0 \in \Omega_{m|2n}$ is called* generic *for a Borel subalgebra $\mathfrak{b} \in \mathcal{B}$ if the $\mathfrak{b}$-highest weight of the corresponding irreducible module is given by*

$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r_{\mathfrak{b}}$, *where* $r_{\mathfrak{b}}$ *is defined in Definition 6.4.5. Otherwise,* $\underline{\lambda}_0$ *is called nongeneric.*

6.4.2 A formula for nongeneric highest weights

In this subsection, we give a general formula for nongeneric highest weights. Let $\underline{\lambda}_0 \in \Omega_{m|2n}$. Recall that a highest weight $\underline{\lambda}_{\mathfrak{b}}$ for $\mathfrak{b} \in \mathcal{B}$ is obtained by subtracting a subset (potentially the full set if $\underline{\lambda}_0$ is generic) of $\mathcal{G}_{\mathfrak{b}}$ from $\underline{\lambda}_0$. Thus we give the following definition.

Definition 6.4.10. *Let* $\underline{\lambda}_0 \in \Omega_{m|2n}$. *Let* $\mathfrak{b} \in \mathcal{B}$ *and let* $\underline{\lambda}_{\mathfrak{b}}$ *be the highest weight of the corresponding module with respect to* $\mathfrak{b}$*. Then the term*

$$r(\underline{\lambda}, \mathfrak{b}) \coloneqq \underline{\lambda}_0 - \underline{\lambda}_{\mathfrak{b}}$$

is the sum of the roots that are subtracted from $\underline{\lambda}_0$ *to produce* $\underline{\lambda}_{\mathfrak{b}}$.

Remark 6.4.11. Notice that if $\underline{\lambda}_0 \in \Omega_{m|2n}$ is *nongeneric* for some $\mathfrak{b} \in \mathcal{B}$, then $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - \sum_{\alpha \in \mathcal{N}_{\mathfrak{b}}} \alpha$ for some proper subset $\mathcal{N}_{\mathfrak{b}}$ of $\mathcal{G}_{\mathfrak{b}}$. Therefore,

$$r(\underline{\lambda}, \mathfrak{b}) \coloneqq \begin{cases} r_{\mathfrak{b}} & \text{if } \underline{\lambda}_{\mathfrak{b}} \text{ is generic for } \mathfrak{b}, \\ \sum_{\alpha \in \mathcal{N}_{\mathfrak{b}}} \alpha & \text{if } \underline{\lambda}_{\mathfrak{b}} \text{ is nongeneric for } \mathfrak{b}. \end{cases}$$

Our next goal is to derive a formula for $r(\underline{\lambda}, \mathfrak{b})$. To achieve the goal, we first give two definitions.

Definition 6.4.12. *Let* $\mathfrak{b} \in \mathcal{B}$ *and* $\lambda = (2\lambda_1, \ldots, 2\lambda_{m+2n}) \in \mathcal{H}(m, 2n)$. *If* $2\lambda_m < \ell_m$, *then let* $I_{\lambda,\mathfrak{b}}$ *be the least index* I *for which*

$$\ell_I > 2\lambda_I.$$

The index $I_{\lambda,\mathfrak{b}}$ is an important index that shows up frequently in the rest of this chapter. In fact, when we compute highest weights with respect to different $\mathfrak{b} \in \mathcal{B}$, $I_{\lambda,\mathfrak{b}}$ gives the index where we can identify if we should or should not subtract odd roots from the previous highest weights. That is, the highest weight only changes when we apply an odd reflection $r_{\delta_i - \epsilon_j}$ for some $1 \leq i \leq 2n$ and $j \leq I_{\lambda,\mathfrak{b}}$.

Example 6.4.13. Retain the setup in Example 6.4.7, where we have computed that

$$\left(\ell_{1,\mathfrak{b}}, \ell_{2,\mathfrak{b}}, \ell_{3,\mathfrak{b}}, \ell_{4,\mathfrak{b}}, \ell_{5,\mathfrak{b}}, \ell_{6,\mathfrak{b}}\right) = (0,0,4,4,8,8)\,.$$

Thus if $\underline{\lambda}_0 = -\left(6,4,2,2,0,0 \mid 0_{1\times 8}\right)$, then $I_{\lambda,\mathfrak{b}} = 3$. ♠

Definition 6.4.14. *Let $\mathfrak{b}$ be a Borel subalgebra corresponding to a decreasing $\delta\epsilon$ sequence. Let $\underline{\lambda}_0$ be a highest weight for $\mathfrak{b}_{\text{op}}$. Define*

$$\ell_{\underline{\lambda},i} = \begin{cases} \ell_{i,\mathfrak{b}} & \text{if } 1 \leq i < I_{\lambda,\mathfrak{b}}, \\ 2\lambda_i & \text{if } I_{\lambda,\mathfrak{b}} \leq i \leq m. \end{cases}$$

Then for each $1 \leq k \leq 2n$, set $j_{\underline{\lambda},k}$ to be the number of indices i such that $\ell_{\underline{\lambda},i} \geq k$.

Example 6.4.15. Let $\mathfrak{b} = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\epsilon_4\epsilon_3\delta_4\delta_3\delta_2\delta_1\epsilon_2\epsilon_1$ as in Example 6.4.7. We have computed that

$$\left(\ell_{1,\mathfrak{b}}, \ell_{2,\mathfrak{b}}, \ell_{3,\mathfrak{b}}, \ell_{4,\mathfrak{b}}, \ell_{5,\mathfrak{b}}, \ell_{6,\mathfrak{b}}\right) = (0,0,4,4,8,8)\,,$$

which implies that

$$\left(\ell_{\underline{\lambda}_{\mathfrak{b}},1}, \ell_{\underline{\lambda}_{\mathfrak{b}},2}, \ell_{\underline{\lambda}_{\mathfrak{b}},3}, \ell_{\underline{\lambda}_{\mathfrak{b}},4}, \ell_{\underline{\lambda}_{\mathfrak{b}},5}, \ell_{\underline{\lambda}_{\mathfrak{b}},6}\right) = (0,0,2,2,0,0)\,,$$

and in turn,

$$\left(j_{\underline{\lambda}_{\mathfrak{b}},1}, j_{\underline{\lambda}_{\mathfrak{b}},2}, j_{\underline{\lambda}_{\mathfrak{b}},3}, j_{\underline{\lambda}_{\mathfrak{b}},4}, j_{\underline{\lambda}_{\mathfrak{b}},5}, j_{\underline{\lambda}_{\mathfrak{b}},6}\right) = (2,2,0,0,0,0)\,. \qquad \spadesuit$$

Lemma 6.4.16. *The definition of $\ell_{\underline{\lambda},i}$ is equivalent to*

$$\ell_{\underline{\lambda},i} = \begin{cases} \ell_{i,\mathfrak{b}} & \text{if } \ell_{i,\mathfrak{b}} \leq 2\lambda_i, \\ 2\lambda_i & \text{if } \ell_{i,\mathfrak{b}} > 2\lambda_i. \end{cases}$$

Thus, we have that $\ell_{\underline{\lambda},i} = \min\{\ell_{i,\mathfrak{b}}, 2\lambda_i\}$ for all $1 \leq i \leq m$.

Proof. The result follows from the fact that for all $i < I_{\lambda,\mathfrak{b}}$, we have $\ell_{i,\mathfrak{b}} \leq 2\lambda_{i,\mathfrak{b}}$ and for all $j > I_{\lambda,\mathfrak{b}}$ we have

$$\ell_{j,\mathfrak{b}} \geq \ell_{I_{\lambda,\mathfrak{b}},\mathfrak{b}} > 2\lambda_{I_{\lambda,\mathfrak{b}}} \geq 2\lambda_j. \quad \square$$

Proposition 6.4.17. *Let $\mathfrak{b}$ be a Borel subalgebra corresponding to a decreasing $\delta\epsilon$ sequence. Let $\underline{\lambda}_0$ be a highest weight for $\mathfrak{b}_{\mathrm{op}}$. Then*

$$r(\underline{\lambda}, \mathfrak{b}) = -\sum_{i=1}^{m} \ell_{\underline{\lambda},i}\epsilon_i + \sum_{k=1}^{2n} j_{\underline{\lambda},k}\delta_k. \tag{6.4.1}$$

In particular, $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b})$ is the $\mathfrak{b}$-highest weight of the module corresponding to λ, for any $\lambda \in \mathcal{H}(m|2n)$.

We first give a detailed example to illustrate that $r(\underline{\lambda}, \mathfrak{b})$ is precisely the sum of odd roots we *should* subtract from $\underline{\lambda}_0$ to obtain $\underline{\lambda}_{\mathfrak{b}}$.

Example 6.4.18. Let $\mathfrak{b}_{\mathrm{op}} = \delta_8 \ldots \delta_1\epsilon_6 \ldots \epsilon_1$ be the opposite standard Borel subalgebra of $\mathfrak{gl}(6|8)$. Let $\mathfrak{b} = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\epsilon_4\epsilon_3\delta_4\delta_3\delta_2\delta_1\epsilon_2\epsilon_1$ and $\underline{\lambda}_0 = -\left(6,4,2,2,0,0 \mid 0_{1\times 8}\right)$ as in Example 6.4.13.

To illustrate in detail how the highest weight changes with respect to different Borel subalgebras, we first consider moving ϵ_6 to the leftmost of the $\delta\epsilon$-sequence by consecutively apply $r_{\delta_1-\epsilon_6}, \ldots, r_{\delta_8-\epsilon_6}$ to $\mathfrak{b}_{\rm op}$. In particular, we have that $(\underline{\lambda}_0, \delta_i - \epsilon_6) = 0$ for all $1 \leq i \leq 8$. Thus the resulting highest weight is still $\underline{\lambda}_0$. The same is true for moving ϵ_5, that is, we have that $(\underline{\lambda}_0, \delta_i - \epsilon_5) = 0$ for all $1 \leq i \leq 8$. Now consider the next few intermediate Borel subalgebras. Let

$$\mathfrak{b}_1 = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\delta_4\delta_3\delta_2\delta_1\epsilon_4\epsilon_3\epsilon_2\epsilon_1$$

$$\mathfrak{b}_2 = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\delta_4\delta_3\delta_2\epsilon_4\delta_1\epsilon_3\epsilon_2\epsilon_1$$

$$\mathfrak{b}_3 = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\delta_4\delta_3\epsilon_4\delta_2\delta_1\epsilon_3\epsilon_2\epsilon_1$$

$$\mathfrak{b}_4 = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\delta_5\epsilon_4\delta_4\delta_3\delta_2\delta_1\epsilon_3\epsilon_2\epsilon_1$$

That is, $\mathfrak{b}_2 = r_{\delta_1-\epsilon_4}(\mathfrak{b}_1)$, $\mathfrak{b}_3 = r_{\delta_2-\epsilon_4}(\mathfrak{b}_2)$ and $\mathfrak{b}_4 = r_{\delta_4-\epsilon_4} r_{\delta_3-\epsilon_4}(\mathfrak{b}_3)$. Therefore, we have

$$\mathfrak{b}_4 = r_{\delta_8-\epsilon_6} \ldots r_{\delta_1-\epsilon_6} r_{\delta_8-\epsilon_5} \ldots r_{\delta_1-\epsilon_5}(\mathfrak{b}_{\rm op}).$$

Recall $\underline{\lambda}_{\mathfrak{b}_1} = \underline{\lambda}_0$. Moreover, we have that $\mathfrak{b}_2 = r_{\delta_1-\epsilon_4}(\mathfrak{b}_1)$ and $\left(\underline{\lambda}_{\mathfrak{b}_1}, \delta_1 - \epsilon_4\right) = 2 \neq 0$, which by Corollary 2.2.5 implies that

$$\underline{\lambda}_{\mathfrak{b}_2} = \underline{\lambda}_{\mathfrak{b}_1} - (\delta_1 - \epsilon_4) = -\left(6, 4, 2, 1, 0, 0 \mid 1, 0_{1\times 7}\right).$$

Similarly, we have that $\mathfrak{b}_3 = r_{\delta_2-\epsilon_4}(\mathfrak{b}_2)$ and $\left(\underline{\lambda}_{\mathfrak{b}_2}, \delta_2 - \epsilon_4\right) = 2 \neq 0$ which implies that

$$\underline{\lambda}_{\mathfrak{b}_3} = \underline{\lambda}_{\mathfrak{b}_2} - (\delta_2 - \epsilon_4) = -\left(6, 4, 2, 0, 0, 0 \mid 1, , 1, 0_{1\times 6}\right).$$

Furthermore, we have $\mathfrak{b}_4 = r_{\delta_4-\epsilon_4} r_{\delta_3-\epsilon_4}(\mathfrak{b}_3)$ and $\left(\underline{\lambda}_{\mathfrak{b}_3}, \delta_i - \epsilon_4\right) = 0$ for $i = 3, 4$. Thus

$\underline{\lambda}_{\mathfrak{b}_4} = \underline{\lambda}_{\mathfrak{b}_3}$. Moreover, we have that

$$\mathfrak{b} = r_{\delta_4-\epsilon_3} r_{\delta_3-\epsilon_3} r_{\delta_2-\epsilon_3} r_{\delta_1-\epsilon_3}(\mathfrak{b}_4)$$

and by a similar argument we have that

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_{\mathfrak{b}_4} - (\delta_1 - \epsilon_3) - (\delta_1 - \epsilon_3) = \underline{\lambda}_{\mathfrak{b}} = -(6,4,0,0,0,0|2,2,0_{1\times 6}).$$

In particular, the sum of odd roots we subtract from $\underline{\lambda}_0$ is precisely

$$r(\underline{\lambda}_{\mathfrak{b}}, \mathfrak{b}) = -2\epsilon_3 - 2\epsilon_4 + 2\delta_1 + 2\delta_2. \qquad \spadesuit$$

Proof of Proposition 6.4.17. To show that $r(\underline{\lambda}, \mathfrak{b})$ is of the claimed form, it is equivalently to showing that $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b})$. Since the $\delta\epsilon$ sequence is decreasing, we must have

$$\ell_{1,\mathfrak{b}} \leq \ell_{2,\mathfrak{b}} \leq \ell_{3,\mathfrak{b}} \leq \cdots \leq \ell_{m,\mathfrak{b}}.$$

Since $\underline{\lambda}_0 = -(2\lambda_1, \cdots, 2\lambda_m, \mu_1, \mu_1, \cdots, \mu_n, \mu_n)$ corresponds to a hook diagram λ, we have

$$2\lambda_1 \geq 2\lambda_2 \geq \cdots \geq 2\lambda_m.$$

The generic case occurs when $\ell_{i,\mathfrak{b}} \leq 2\lambda_i$ for all $1 \leq i \leq m$, which is by the preceding inequalities equivalent to the condition that $\ell_{m,\mathfrak{b}} \leq 2\lambda_m$. Thus $\ell_{\underline{\lambda},i} = \ell_{i,\mathfrak{b}}$ for all $1 \leq i \leq m$, and similarly, $j_{\underline{\lambda},k} = j_{k,\mathfrak{b}}$ for all $1 \leq k \leq 2n$. Therefore, $r(\underline{\lambda}, \mathfrak{b}) = r_{\mathfrak{b}}$ and we are done.

Now suppose $\underline{\lambda}_0$ is not generic and let $I = I_{\lambda,\mathfrak{b}}$ be the least index for which $\ell_{I,\mathfrak{b}} > 2\lambda_I$. We now show that $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b})$.

Choose a sequence of Borel subalgebras linking $\mathfrak{b}_{\mathrm{op}}$ to $\mathfrak{b}$ that is strictly decreasing. In terms of $\delta\epsilon$ sequences, this implies that we first put ϵ_m into position, then ϵ_{m-1}, and so forth. Recall from Definition 6.4.5 that the sequence of simple odd roots is then

$$\begin{aligned}
&\delta_1 - \epsilon_m, \delta_2 - \epsilon_m, \cdots, \delta_{\ell_m} - \epsilon_m; \qquad (*)\\
&\delta_1 - \epsilon_{m-1}, \delta_2 - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1}} - \epsilon_{m-1};\\
&\cdots\\
&\delta_1 - \epsilon_1, \delta_2 - \epsilon_1, \cdots, \delta_{\ell_1} - \epsilon_1.
\end{aligned}$$

At each step $\mathfrak{b}_i$ of this sequence, corresponding to applying the reflection r_{α_i}, we use the formula

$$\underline{\lambda}_{\mathfrak{b}_i} = \begin{cases} \underline{\lambda}_{\mathfrak{b}_{i-1}} - \alpha_i & \text{if } (\underline{\lambda}_{\mathfrak{b}_{i-1}}, \alpha_i) \neq 0 \\ \underline{\lambda}_{\mathfrak{b}_{i-1}} & \text{otherwise.} \end{cases} \tag{6.4.2}$$

Note that $(\underline{\lambda}_{\mathfrak{b}'}, \delta_k - \epsilon_i) = 0$ if and only if the coefficients of both ϵ_i and δ_k in $\lambda_{\mathfrak{b}'}$ are zero. Note also that in $\underline{\lambda}_0$, we have $\mu_k = 0$ for all $k > \lambda_m$.

We proceed inductively from m down to 1, following the rows of the list of roots (*), showing that if $\mathfrak{b}_{j+1}$ denotes the Borel subalgebra obtained after completing all reflections up to the end of those in the row corresponding to ϵ_{j+1}, that

$$\underline{\lambda}_{\mathfrak{b}_{j+1}} = \underline{\lambda}_0 - \sum_{i=j+1}^{m} (\delta_1 + \cdots + \delta_{\ell_{\underline{\lambda},i}} - \ell_{\underline{\lambda},i}\epsilon_i). \tag{6.4.3}$$

The base case corresponds to $j = m$, $\mathfrak{b} = \mathfrak{b}_{\mathrm{op}}$, where there is nothing to show.

Suppose we have shown our inductive hypothesis up to ϵ_{j+1}, as in (6.4.3). Let us first determine the coefficients of ϵ_j and of δ_k with $k > 2\lambda_j$.

The coefficient of ϵ_j in $-\underline{\lambda}_{\mathfrak{b}_{j+1}}$ is the same as that of $-\underline{\lambda}_0$; this value is therefore $2\lambda_j$. Let $k > 2\lambda_j \geq 2\lambda_m$. Then the coefficient of δ_k in $\underline{\lambda}_0$ was zero. Moreover, since $2\lambda_j \geq 2\lambda_i \geq \ell_{\underline{\lambda},i}$ for each $j+1 \leq i \leq m$, none of the additional terms in (6.4.3) contribute to the coefficient of δ_k. Thus the coefficient of δ_k in $-\underline{\lambda}_{\mathfrak{b}_{j+1}}$ is 0 for $k > 2\lambda_j$.

As we iterate through each of the roots

$$\delta_1 - \epsilon_j, \delta_2 - \epsilon_j, \cdots, \delta_{\ell_{\lambda,j}} - \epsilon_j$$

we fall into the first case of (6.4.2) because the coefficient of ϵ_j is nonzero at each iteration. Therefore if $\mathfrak{b}'$ denotes the resulting Borel subalgebra, we have

$$\underline{\lambda}_{\mathfrak{b}'} = \underline{\lambda}_{\mathfrak{b}_{j+1}} - (\delta_1 + \cdots + \delta_{\ell_{\lambda,j}} - \ell_{\lambda,j}\epsilon_j).$$

If $\ell_{\underline{\lambda},j} = \ell_{i,\mathfrak{b}}$ then we are done.

Otherwise, $\ell_{\underline{\lambda},j} = 2\lambda_j$ and we infer that the coefficient of ϵ_j in $\lambda_{\mathfrak{b}'}$ is 0, as is the coefficient of δ_k for any $k > \ell_{\underline{\lambda},j} = 2\lambda_j$. Thus for any remaining roots

$$\delta_{\ell_{\underline{\lambda},j}+1} - \epsilon_j, \cdots, \delta_{\ell_j} - \epsilon_j$$

we fall into the second case of (6.4.2), and so

$$\underline{\lambda}_{\mathfrak{b}_j} = \underline{\lambda}_{\mathfrak{b}'} = \underline{\lambda}_0 - \sum_{i=j}^{m}(\delta_1 + \cdots + \delta_{\ell_{\underline{\lambda},i}} - \ell_{\underline{\lambda},i}\epsilon_i).$$

Therefore we may conclude by induction that

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - \sum_{i=1}^{m}(\delta_1 + \cdots + \delta_{\ell_{\underline{\lambda},i}} - \ell_{\underline{\lambda},i}\epsilon_i) = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}),$$

as required. □

We complete this subsection by giving the list of roots contained in $r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})$, that is, the set of roots which we will *not* subtract from λ_0 when computing the highest weight $\underline{\lambda}_{\mathfrak{b}}$.

Lemma 6.4.19. *Let $\mathfrak{b} \in \mathcal{B}$. Let $\underline{\lambda}_0$ be a highest weight with respect to $\mathfrak{b}_{\text{op}}$. Let $I = I_{\lambda,\mathfrak{b}}$ be the least index for which $\ell_{I,\mathfrak{b}} > 2\lambda_I$. Then $r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})$ is the sum of the following roots.*

$$\delta_{2\lambda_m+1} - \epsilon_m, \cdots, \delta_{\ell_{m,\mathfrak{b}}} - \epsilon_m, \tag{**}$$

$$\delta_{2\lambda_{m-1}+1} - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1,\mathfrak{b}}} - \epsilon_{m-1},$$

$$\cdots$$

$$\delta_{2\lambda_I+1} - \epsilon_I, \cdots, \delta_{\ell_{I,\mathfrak{b}}} - \epsilon_I,$$

Proof. This is straightforward from the definition of $I_{\lambda,\mathfrak{b}}, r_{\mathfrak{b}}, r(\underline{\lambda}, \mathfrak{b})$ and the proof of Proposition 6.4.17. □

6.5 The CEP with compatible highest weights

In this section, we first derive some necessary conditions that the affine map $\tau_{\mathfrak{b}} : \mathfrak{h}^* \to \mathbb{C}^{m|n}$ must satisfy in order for the relation

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) := M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) + X_{\mathfrak{b}} \sim \tau_0(\underline{\lambda}_0) \tag{6.5.1}$$

to hold for any $\lambda \in \mathcal{H}(m|2n)$ with even parts. Moreover, we define a class of Borel subalgebras that we call very even, for which every $\underline{\lambda}_0 \in \Omega_{m|2n}$ is compatible.

Then we prove the result for very even Borel subalgebras and prove that, when $\underline{\lambda}_0$ is compatible with (not necessarily very even) $\mathfrak{b}$, the eigenvalue of $D^{\mu}_{m|2n}$ on the irreducible components $\left(V_{m|2n}\right)^*$ is $SP^*_{\mu,\frac{1}{2}} \circ \tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$.

Lemma 6.5.1. *For all $\mathfrak{b} \in \mathcal{B}$, if $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) + X_{\mathfrak{b}}$ such that* (6.5.1) *holds, then we must have*

$$X_{\mathfrak{b}} \sim X_0.$$

Proof. Since the zero weight corresponds to the trivial module, the highest weight $\underline{\lambda}$ of this module with respect to any Borel subalgebra is 0. Thus, we have that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$ implies $M_{\mathfrak{b}}(0) + X_{\mathfrak{b}} \sim M_0(0) + X_0$ which implies $X_{\mathfrak{b}} \sim X_0$. □

Proposition 6.5.2. *For all $\mathfrak{b} \in \mathcal{B}$, if $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) + X_{\mathfrak{b}}$ such that* (6.5.1) *holds, then we must have the identity*

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M_0\underline{\lambda}_0 + X_0 \tag{6.5.2}$$

for all highest weights $\underline{\lambda}_0$ that are generic for $\mathfrak{b}$. Consequently, for all $\mathfrak{b} \in \mathcal{B}$ we must have

$$M_{\mathfrak{b}} \in \mathcal{C} \quad \textit{and} \quad M_{\mathfrak{b}} r_{\mathfrak{b}} = X_{\mathfrak{b}} - X_0.$$

Proof. The idea is as follows: for sufficiently generic $\underline{\lambda}_0$, the identity $M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} \sim M_0\underline{\lambda}_0 + X_0$ must be an equality. More precisely: we call $\underline{\lambda}_0$ generic for $\mathfrak{b}$ if $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r_{\mathfrak{b}}$. This corresponds to the condition that

$$2\lambda_i \geq \ell_i, \quad 1 \leq i \leq m.$$

It follows that the span of the $\mathfrak{b}_{\mathrm{op}}$-highest weights that are generic for $\mathfrak{b}$ is equal to $\mathfrak{a}^*$, the span of the set of all $\mathfrak{b}_{\mathrm{op}}$-highest weights. Since each matrix $M \in \mathcal{C}$ has full rank, it is surjective. Thus we can find a set of $m+n$ linearly independent $\mathfrak{b}_{\mathrm{op}}$-highest weight vectors such that they are generic for $\mathfrak{b}$ and also $M\underline{\lambda}_0$ has all nonzero entries. Then $s\underline{\lambda}_0$ is also generic for any positive integer s. Then for s sufficiently large, there are no pairs (i,j) to which monoidal symmetry could be applied, that is, we can ensure by choosing s large enough that $(s\lambda_i + E_i) + \frac{1}{2}(s\mu_j + F_j) \neq \pm\frac{1}{4}$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$, that is, the equivalence relation in Lemma 6.1.4 can never be used. Ergo, for the highest weights $\underline{\lambda}_0$ in the basis $\mathfrak{b}$, we must have the equality

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M_0\underline{\lambda}_0 + X_0.$$

Since $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r_{\mathfrak{b}}$ we can rewrite this as

$$M_{\mathfrak{b}}\underline{\lambda}_0 + (-M_{\mathfrak{b}}r_{\mathfrak{b}} + X_{\mathfrak{b}}) = M_0\underline{\lambda}_0 + X_0.$$

Since this holds on a basis of $\mathfrak{a}^*$, we conclude that these affine maps from $\mathfrak{a}^*$ to $\mathbb{C}^{m|n}$ are equal. In particular, equality holds for any $\mathfrak{b}_{\mathrm{op}}$-highest weight $\underline{\lambda}_0$, and thus (6.5.2) holds for any $\underline{\lambda}_0$ that is generic for $\mathfrak{b}$.

Furthermore, the equality of the affine maps implies $-M_{\mathfrak{b}}r_{\mathfrak{b}} + X_{\mathfrak{b}} = X_0$ and, by Lemma 6.2.2, that $M_{\mathfrak{b}} - M_0 \in \mathcal{C}$. □

6.5.1 The very even case

In this subsection, we first define a set of Borel subalgebras we call very even, which is a small subset of all Borel subalgebras. However, this set plays a vital role in finding the refined solution to the CEP for $(\mathfrak{g}, \mathfrak{b}, V)$.

Definition 6.5.3. *We call a Borel subalgebra* $\mathfrak{b}$ *and its associated* $\delta\epsilon$ *sequence* very even *if* $\ell_{i,\mathfrak{b}}$ *is even for all* i.

Lemma 6.5.4. *The Borel subalgebra* $\mathfrak{b}$ *is very even if and only if* $j_{2k-1} = j_{2k}$ *for all* $1 \leq k \leq n$.

Proof. By definition, $j_{2k-1} = j_{2k}$ if and only if no ϵ lies in between δ_{2k} and δ_{2k-1} if and only if the Borel subalgebra sequence has the form of $\ldots\delta_{2k}\delta_{2k-1}\ldots$ for all $k \in \{1, \ldots, n\}$ if and only if $\ell_{i,\mathfrak{b}}$ is even for all i. □

Lemma 6.5.5. *If* $\mathfrak{b}$ *is very even and* $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) + X_{\mathfrak{b}}$ *satisfies* (6.5.1)*, then we must have*

$$X_{\mathfrak{b}} = X_0 + \sum_{i=1}^{m} \frac{1}{2}\ell_{i,\mathfrak{b}} e_i - \sum_{k=1}^{n} j_{2k,\mathfrak{b}} e_{m+k}. \tag{6.5.3}$$

Proof. Suppose $\mathfrak{b}$ is very even and let $\underline{\lambda}_0$ be a $\mathfrak{b}_{\mathrm{op}}$-highest weight that is generic for $\mathfrak{b}$. By Proposition 6.5.2, we must have

$$M_{\mathfrak{b}} \in \mathcal{C} \text{ and } M_{\mathfrak{b}} r_{\mathfrak{b}} = X_{\mathfrak{b}} - X_0.$$

Then note that $r_{\mathfrak{b}} = \underline{\lambda}_0 - \underline{\lambda}_{\mathfrak{b}}$ lies in $\mathfrak{a}^*$, the span of highest weights with respect to $\mathfrak{b}_{\mathrm{op}}$, since $\mathfrak{b}$ is very even. It then follows that $M_{\mathfrak{b}} r_{\mathfrak{b}} = M r_{\mathfrak{b}}$ for all $M \in \mathcal{C}$. A direct computation yields the result. □

Thus we have the following proposition.

Proposition 6.5.6. *Let* $\mathfrak{b}$ *be a very even Borel subalgebra. Let* $M_{\mathfrak{b}} \in \mathcal{C}$ *and* $X_{\mathfrak{b}} = M_{\mathfrak{b}} r_{\mathfrak{b}} + X_0$ *as in* (6.5.3)*. Then for any vector* $y \in \mathfrak{a}^*$ *and any* $M_0 \in \mathcal{C}$ *we have*

$$M_{\mathfrak{b}}(y - r_{\mathfrak{b}}) + X_{\mathfrak{b}} = M_0 y + X_0.$$

Proof. Note that since $y \in \mathfrak{a}^*$, we have that $M_{\mathfrak{b}} y = M_0 y$ for all $M_{\mathfrak{b}}, M_0 \in \mathcal{C}$. Thus by Proposition 6.5.2 we have that

$$M_{\mathfrak{b}}(y - r_{\mathfrak{b}}) + X_{\mathfrak{b}} = M_{\mathfrak{b}} y - (X_{\mathfrak{b}} - X_0) + X_{\mathfrak{b}} = M_0 y + X_0.$$

which completes the proof. □

We now turn to the nongeneric case. Recall that the sum of roots $r(\underline{\lambda}, \mathfrak{b})$ defined in Equation (6.4.1) satisfies $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b})$.

We first prove our key technical result, which gives a collection of terms that are equivalent to $\tau_0(\underline{\lambda}_0)$.

Proposition 6.5.7. *Suppose* $\mathfrak{b}$ *is very even and that* $\underline{\lambda}_0 \in \Omega_{m|2n}$. *Then*

$$\lambda_0 + r_{\mathfrak{b}} \in \mathfrak{a}^*,$$

and

$$M_0 \underline{\lambda}_0 + X_0 \sim M_0(\lambda_0 + r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})) + X_0.$$

Proof. First notice that both $\underline{\lambda}_0$ and $r_{\mathfrak{b}}$ are in $\mathfrak{a}^*$ which is clear from the fact that we are in the very even case. By Lemma 6.4.19, $r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})$ is the sum of the following roots, which we showed in the proof of Lemma 6.5.5:

$$\begin{aligned}
&\delta_{2\lambda_m+1} - \epsilon_m, \cdots, \delta_{\ell_{m,\mathfrak{b}}} - \epsilon_m,\\
&\delta_{2\lambda_{m-1}+1} - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1,\mathfrak{b}}} - \epsilon_{m-1},\\
&\cdots\\
&\delta_{2\lambda_I+1} - \epsilon_I, \cdots, \delta_{\ell_{I,\mathfrak{b}}} - \epsilon_I,
\end{aligned}$$

where I is the least index for which $\ell_{I,\mathfrak{b}} > 2\lambda_I$. Suppose that $\mathfrak{b}$ is very even, so that $\ell_{I,\mathfrak{b}} \geq 2\lambda_I + 2$. Then we have

$$\lambda_m + 1 \leq \lambda_{m-1} + 1 \leq \cdots \leq \lambda_I + 1 \leq \frac{1}{2}\ell_{I,\mathfrak{b}} \leq \frac{1}{2}\ell_{I+1,\mathfrak{b}} \leq \cdots \leq \frac{1}{2}\ell_{m,\mathfrak{b}}.$$

That is, the intervals $(\lambda_i + 1, \ell_{i,\mathfrak{b}})$ are nested, as i decreases.

We show that we can transform $\Lambda_{m+1} := M_0\underline{\lambda}_0 + X_0$, by a sequence of monoidal symmetries determined by (**), into the final result

$$\Lambda_{I-1} = M_0(\underline{\lambda}_0 + r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})) + X_0.$$

We proceed by induction on i from m to I on the rows of (**). Let

$$r_{b,\lambda,j} = \sum_{k=2\lambda_j+1}^{\ell_{j,\mathfrak{b}}} \delta_k - (\ell_{j,\mathfrak{b}} - 2\lambda_j)\epsilon_j$$

denote the sum of the roots on the row of (**) corresponding to ϵ_j. Set

$$\Lambda_{i+1} = M_0(\underline{\lambda}_0 + \sum_{j=i+1}^{m} r_{b,\lambda,j}) + X_0,$$

which is the partial sum obtained after completing the row indexed by $i+1$.

Our inductive hypothesis is that

$$\Lambda_{i+1} \sim \Lambda_i,$$

that the coefficient of e_i in Λ_{i+1} is $\lambda_i + E_i$ and that the coefficients of e_{m+j} in Λ_{i+1} with $\lambda_i + 1 \leq j \leq \frac{1}{2}\ell_{i,\mathfrak{b}}$ are given by $F_j - (m-i)$.

The base case is that $i = m$, so equivalence is a tautology and the statement about the coefficients is true. Suppose it holds for $i+1$. Let us prove the statement for i. Note that

$$\begin{aligned}
\Lambda_i &= \Lambda_{i+1} + M_0 r_{b,\lambda,i} \\
&= \Lambda_{i+1} + M_0 \left(\left(\sum_{t=2\lambda_i+1}^{\ell_{i,\mathfrak{b}}} \delta_t \right) - (\ell_{i,\mathfrak{b}} - 2\lambda_i)\epsilon_i \right) \\
&= \Lambda_{i+1} + M_0 \left(\sum_{k=\lambda_i+1}^{\ell_{i,\mathfrak{b}}/2} (\delta_{2k-1} + \delta_{2k} - 2\epsilon_i) \right) \\
&= \Lambda_{i+1} + \sum_{k=\lambda_i+1}^{\ell_{i,\mathfrak{b}}/2} (e_i - e_{m+k})
\end{aligned}$$

Recall that monoidal symmetry defined in Lemma 6.1.4 gives us that $y \sim y + e_i - e_{m+k}$ if $y_i + \frac{1}{2} y_{m+k} = -\frac{1}{4}$, and that

$$E_i + \frac{1}{2} F_j = \frac{1}{2}(m-i) - j + \frac{3}{4}.$$

Suppose we have shown for some K, $\lambda_i + 1 \leq K < \ell_i/2$, that Λ_{i+1} is equivalent to

$$\Lambda_{i+1,K-1} = \Lambda_{i+1} + \sum_{k=\lambda_i+1}^{K-1} (e_i - e_{m+k}).$$

We now show that $\Lambda_{i+1,K-1} \sim \Lambda_{i+1,K}$. Then by induction as K runs from $\lambda_i + 1$ to $\ell_{i,\mathfrak{b}}/2$, we will conclude that $\Lambda_i \sim \Lambda_{i+1}$.

Using the outer induction hypothesis, as well as the formula for $\Lambda_{i+1,K-1}$, we compute that the coefficient of e_i in $\Lambda_{i+1,K-1}$ is $\lambda_i + E_i + (K - \lambda_i - 1)$ and the coefficient of e_{m+K} in $\Lambda_{i+1,K-1}$ is $F_K - (m-i)$. We now verify the conditions for applying monoidal symmetry. We compute

$$\begin{aligned}
&\lambda_i + E_i + (K - \lambda_i - 1) + \frac{1}{2}\left(F_K - (m-i)\right)) \\
=& \left(E_i + \frac{1}{2}F_K\right) + K - 1 - \frac{1}{2}(m-i) \\
=& \left(\frac{1}{2}(m-i) - K + \frac{3}{4}\right) + K - 1 - \frac{1}{2}(m-i) = -\frac{1}{4}
\end{aligned}$$

Therefore monoidal symmetry may be applied and we conclude that

$$\Lambda_{i+1,K-1} \sim \Lambda_{i+1,K-1} + e_i - e_{m+K} = \Lambda_{i+1,K},$$

as required. Therefore we have deduced by induction that

$$\Lambda_{i+1} \sim \Lambda_{i+1,\ell_i/2} = \Lambda_i.$$

Finally, if $i-1 \geq I$, we compute the coefficients of Λ_i, to complete the outer induction.

The coefficient of e_{i-1} in Λ_i is the same as the coefficient of e_{i-1} in $M_0\underline{\lambda}_0 + X_0$, which is $\lambda_{i-1} + E_{i-1}$, because we have added no roots involving ϵ_{i-1}.

Since $i-1 \geq I$ and $\ell_{i-1,\mathfrak{b}}$ is even, we have $\lambda_i + 1 \leq \lambda_{i-1} + 1 \leq \frac{1}{2}\ell_{i-1,\mathfrak{b}} \leq \frac{1}{2}\ell_{i,\mathfrak{b}}$. Therefore the coefficient of e_{m+j}, for all $\lambda_{i-1} + 1 \leq j \leq \frac{1}{2}\ell_{i-1,\mathfrak{b}}$ was $F_j - (m-i)$ in Λ_{i+1} and is thus $F_j - (m-i) - 1$ in Λ_i, since we subtracted each e_{m+j} exactly once in the course of computing Λ_i. The induction is complete. □

Using the proposition, we now prove the main result for this section, which is, for the very even case, we find an affine function $\tau_{\mathfrak{b}}$ such that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$. In particular, we have the following theorem.

Theorem 6.5.8. *Let $\mathfrak{b}$ be a very even Borel subalgebra in $\mathcal{B}$. For any $M_{\mathfrak{b}} \in \mathcal{C}$, and* *for any* $\underline{\lambda}_0 \in \Omega_{m|2n}$*, define*

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}}$$

where $X_{\mathfrak{b}}$ *is defined in* (6.5.3)*. Then*

$$SP^*_{\mu,\frac{1}{2}} \circ \tau_{\mathfrak{b}}\left(\underline{\lambda}_{\mathfrak{b}}\right) = SP^*_{\mu,\frac{1}{2}} \circ \tau_0\left(\underline{\lambda}_0\right)$$

where $SP^*_{\mu,\frac{1}{2}}$ *is the interpolation super Jack polynomial from Remark 4.4.5. Consequently, the eigenvalue of the Capelli operator* D^μ *on the irreducible component* $\left(V^\lambda_{m|2n}\right)^*$ *with highest weight* $\underline{\lambda}_{\mathfrak{b}}$ *is equal to* $SP^*_{\mu,\frac{1}{2}}\left(\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})\right)$*.*

Proof. It suffices to show $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$. We have that

$$\begin{aligned}
M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} &= M_{\mathfrak{b}}(\underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b})) + X_{\mathfrak{b}} \\
&= M_{\mathfrak{b}}(\underline{\lambda}_0 + (r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})) - r_{\mathfrak{b}}) + X_{\mathfrak{b}} \\
&= M_0(\underline{\lambda}_0 + r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})) + X_0
\end{aligned}$$

where this last equality follows from Proposition 6.5.6, applied with $y = \underline{\lambda}_0 + r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})$, which lies in $\mathfrak{a}^*$. Finally, $M_0\left(\underline{\lambda}_0 + r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})\right) + X_0 \sim M_0\underline{\lambda}_0 + X_0$ by Proposition 6.5.7. □

6.5.2 The non-very-even cases

Our ultimate goal is to extend the results of the previous section for the very even case as much as possible to the not very even case. The idea is that for an arbitrary decreasing $\delta\epsilon$ sequence, each ϵ_i is swapped with either an even or an odd number of δs. If some ϵ_i is swapped with an odd number of δs, the corresponding $\delta\epsilon$ sequence comes from some $\delta\epsilon$ sequences such that these ϵ_is are swapped with an even number of δs.

Thus understanding very even cases helps us understand the most general cases.

Let $\mathfrak{b}$ be a Borel subalgebra corresponding to an arbitrary decreasing $\delta\epsilon$ sequence. Let $\mathfrak{b}_e$ be the very even Borel subalgebra that corresponds to the $\ell_{i,\mathfrak{b}_e}$-sequence given by

$$\ell_{i,\mathfrak{b}_e} = 2\left\lfloor \frac{1}{2}\ell_{i,\mathfrak{b}_e} \right\rfloor$$

for all $1 \leq i \leq m$. That is, $\mathfrak{b}_e$ is a very even Borel subalgebra and the odd reflections that take $\mathfrak{b}_e$ to $\mathfrak{b}$ are of the form $\alpha = \delta_{2k-1} - \epsilon_i$, where the values k may be repeated but the values i are distinct.

For each $1 < k \leq n$, the indices of the decreasing $\delta\epsilon$ sequence corresponding to $\mathfrak{b}$, between δ_{2k} and δ_{2k-1}, are in the form

$$\cdots \delta_{2k}\epsilon_{m-j_{2k}}\epsilon_{m-j_{2k}-1}\cdots\epsilon_{m-j_{2k-1}+1}\delta_{2k-1}\cdots$$

because j_{2k} counts the number of ϵs to the left of δ_{2k}, for example. Thus δ_{2k} and δ_{2k-1} are adjacent when $j_{2k-1} = j_{2k}$, and otherwise are separated by the $j_{2k-1} - j_{2k}$ adjacent ϵ terms $\epsilon_{m-j_{2k}}, \epsilon_{m-j_{2k}-1}, \cdots, \epsilon_{m-j_{2k-1}+1}$. Furthermore, for each ϵ_i between δ_{2k} and δ_{2k-1}, we have $\ell_i = 2k-1$.

Definition 6.5.9. *Let $\mathfrak{b}$ be as above. For each $1 \leq k \leq n$, let*

$$T_{\mathfrak{b}} = \{1 \leq k \leq n \mid j_{2k} < j_{2k-1}\}.$$

Define for each $k \in T_{\mathfrak{b}}$

$$r_{odd,\mathfrak{b},k} = (j_{2k-1} - j_{2k})\delta_{2k-1} - \sum_{i=m-j_{2k-1}+1}^{m-j_{2k}} \epsilon_i. \tag{6.5.4}$$

Example 6.5.10. Set $\mathfrak{b}$ and $\underline{\lambda}_{\mathfrak{b}}$ as in Example 6.4.18. Let $\mathfrak{b}_1 = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\epsilon_4\delta_5\epsilon_3\delta_4\epsilon_2\epsilon_1\delta_3\delta_2\delta_1$. That is, $\mathfrak{b}_1 = r_{\delta_3-\epsilon_1}r_{\delta_2-\epsilon_1}r_{\delta_1-\epsilon_1}r_{\delta_3-\epsilon_2}r_{\delta_2-\epsilon_2}r_{\delta_1-\epsilon_2}r_{\delta_5-\epsilon_3}(\mathfrak{b})$. Then we have $T_{\mathfrak{b}_1} = \{2,3\}$, in particular, $j_3 - j_4 = 2, j_5 - j_6 = 1$. ♠

Note that the list of roots whose sum is $r_{odd,\mathfrak{b},k}$ is precisely

$$\{\delta_{2k-1} - \epsilon_{m-j_{2k}}, \delta_{2k-1} - \epsilon_{m-j_{2k}-1}, \ldots, \delta_{2k-1} - \epsilon_{m-j_{2k-1}+1}\}.$$

That is, the associated odd reflections are those taking

$$\cdots\delta_{2k}\delta_{2k-1}\epsilon_{m-j_{2k}}\epsilon_{m-j_{2k}-1}\cdots\epsilon_{m-j_{2k-1}+1}\delta_{2k-1}\cdots$$

to

$$\cdots\delta_{2k}\epsilon_{m-j_{2k}}\epsilon_{m-j_{2k}-1}\cdots\epsilon_{m-j_{2k-1}+1}\delta_{2k-1}\cdots.$$

Lemma 6.5.11. *Set* $r_{odd,\mathfrak{b}} = \sum_{k\in T_{\mathfrak{b}}} r_{odd,\mathfrak{b},k}$. *With notation as above,* $r_{\mathfrak{b}} = r_{\mathfrak{b}_e} + r_{odd,\mathfrak{b}}$.

Proof. The result follows from the fact $r_{odd,\mathfrak{b}}$ is the sum of odd roots whose associated odd reflections are those taking $\mathfrak{b}_e$ to $\mathfrak{b}$. □

Set

$$\mathcal{C}_{\mathfrak{b}} = \{M \in \mathcal{C} \mid \forall k \in T_{\mathfrak{b}}, M r_{odd,\mathfrak{b},k} = 0\}. \tag{6.5.5}$$

Theorem 6.5.12. *Set* $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$. *The set* $\mathcal{C}_{\mathfrak{b}}$ *is a nonempty subset of* $\mathcal{C}$ *and for all* $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}$ *and any* $M_0 \in \mathcal{C}$ *we have*

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M_0\underline{\lambda}_0 + X_0$$

for all $\underline{\lambda}_0 \in \Omega_{m|2n}$ *that are generic for* $\mathfrak{b}$.

Remark 6.5.13. Notice that the choice of sthis subset $\mathcal{C}_{\mathfrak{b}}$ of $\mathcal{C}$ depends on the choice of vector $X_{\mathfrak{b}}$.

Proof of Theorem 6.5.12. Since $\underline{\lambda}_0$ is generic for $\mathfrak{b}$, it is also generic for $\mathfrak{b}_e$ since $\ell_{i,\mathfrak{b}_e} \leq \ell_{i,\mathfrak{b}}$ for all i. Therefore since the theorem is proven for the very even case we may apply Proposition 6.5.2 to infer that

$$M\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e} = M\underline{\lambda}_0 + X_0$$

for any $M \in \mathcal{C}$. By the lemma, $\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_{\mathfrak{b}_e} + r_{odd,\mathfrak{b}}$, and by hypothesis $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$. Therefore if M further satisfies

$$Mr_{odd,\mathfrak{b},k} = 0 \qquad \forall k \in T_{\mathfrak{b}} \tag{6.5.6}$$

then $Mr_{odd,\mathfrak{b}} = 0$ and

$$M\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}_e} = M(\underline{\lambda}_{\mathfrak{b}_e} - r_{odd,\mathfrak{b}}) + X_{\mathfrak{b}_e}.$$

Thus by the preceding,

$$M\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} = M(\underline{\lambda}_{\mathfrak{b}_e} - r_{odd,\mathfrak{b}}) + X_{\mathfrak{b}_e} = M_0\underline{\lambda}_0 + X_0,$$

as required.

To see that the set of conditions defined by (6.5.6) is nonempty, write

$$M = \mathcal{M} + R = \begin{bmatrix} -\frac{1}{2}I_m & 0_{m\times 2n} \\ 0_{n\times m} & D \end{bmatrix} + \begin{bmatrix} 0_{(m+n)\times m} & A \end{bmatrix}$$

as in (6.2.1) and (6.2.2). To show that $Mr_{odd,\mathfrak{b},k} = 0$, fix k in $T_\mathfrak{b}$. The $(2k-1)$st column of an arbitrary matrix M in C has the form $(a_1, \cdots, a_m|b_1, \cdots, b_k - 1/2, \cdots b_n)$. Let $w = (j_{2k-1} - j_{2k}) \neq 0$. Thus

$$(Mr_{odd,\mathfrak{b},k})_i = \begin{cases} a_i w \text{ if } 1 \leq i \leq m \text{ is not in the set } \{m - j_{2k-1} + 1, \ldots, m - j_{2k}\} \\ \frac{1}{2} + a_i w \text{ if } 1 \leq i \leq m \text{ is in the set } \{m - j_{2k-1} + 1, \ldots, m - j_{2k}\} \\ b_{i-m} w \text{ if } i > m \text{ and } i \neq m + k \\ (b_k - \frac{1}{2}) w \text{ if } i = m + k. \end{cases}$$

Therefore setting each of these equal to zero fully determines the $2k-1$ column of M (and thus, the $2k$th column). Specifically, for each k in $T_\mathfrak{b}$, the $(m+2k-1)$th column of a matrix in $\mathcal{C}_\mathfrak{b}$ has nonzero entries only in rows $i = m - j_{2k-1}, \ldots, m - j_{2k}$ and these entries are all equal to $\frac{1}{2}(j_{2k-1} - j_{2k})$ and its $(m+2k)$th column has the negative of these entries as well as -1 in row $m+k$.

For each $k \in T_\mathfrak{b}$, the equation $Mr_{odd,\mathfrak{b},k} = 0$ can always be solved by choosing values for the $(2k-1)$th column of A. Since each condition from $k \in T_\mathfrak{b}$ pertains to a different odd-index column, this system is consistent and the set of solutions $\mathcal{C}_\mathfrak{b}$ is nonempty. □

Recall that in the very even case, we defined $r_\mathfrak{b}$ and $r(\underline{\lambda}, \mathfrak{b})$ where $r_\mathfrak{b}$ corresponds to generic highest weights and $r(\underline{\lambda}, \mathfrak{b})$ helps us understand non-generic highest weights, that is, the set of roots we *should* subtract. The term $r_{odd,\mathfrak{b}}$ (or $r_{odd,\mathfrak{b},k}$) plays a similar role as $r_\mathfrak{b}$. We then need to find a similar definition for $r(\underline{\lambda}, \mathfrak{b})$ in order to understand all possible highest weights.

Suppose that in the Borel sequence for $\mathfrak{b}$, there exists a sequence of at least two adjacent ϵs between δ_{2k} and δ_{2k-1}. Since j_{2k} is the number of ϵ to the left of δ_{2k}, we deduce that $\epsilon_m, \cdots, \epsilon_{m-j_{2k}+1}$ are precisely the ϵs to the left of δ_{2k}, and $\epsilon_{m-j_{2k}}$ is the first ϵ to the right of δ_{2k}. Similarly, the index $m - j_{2k-1} + 1$ refers to the first ϵ to the left of δ_{2k-1}. That is, our Borel algebra sequence contains

$$\cdots \delta_{2k}\epsilon_{m-j_{2k}} \cdots \epsilon_{m-j_{2k-1}+1}\delta_{2k-1}.$$

Now suppose that $\underline{\lambda}_0$ is not generic for $\mathfrak{b}$. Note that since the sequence of λ_i is decreasing, we have that

$$\lambda_{m-j_{2k}} \leq \lambda_{m-j_{2k}-1} \leq \cdots \leq \lambda_{m-j_{2k-1}+1}.$$

On the other hand, the value of ℓ_i, for each $m - j_{2k} \leq i \leq m - j_{2k-1} + 1$, is exactly $2k-1$. That is, depending on the relation among $2k-1, 2\lambda_{m-j_{2k}}$ and $2\lambda_{m-j_{2k-1}+1}$, we should subtract all, none or partial roots from the set

$$\{\delta_{2k-1} - \epsilon_{m-j_{2k}}, \ldots, \delta_{2k-1} - \epsilon_{m-j_{2k-1}+1}\}.$$

These observations are the bases for the following definitions.

Definition 6.5.14. *We say that a highest weight* $\underline{\lambda}_0 \in \Omega_{m|2n}$ *(or the corresponding hook Young diagram* λ*) is* compatible *with* $\mathfrak{b} \in B$ *if it is either generic for* $\mathfrak{b}$*, or for each* $k \in T_{\mathfrak{b}}$*, either* $2\lambda_{m-j_{2k}} > 2k-1$ *or* $2\lambda_{m-j_{2k-1}+1} < 2k-1$*.*

Definition 6.5.15. *We call a Borel subalgebra* $\mathfrak{b}$ relatively even *if there is at most one* ϵ *between* $\delta_{j_{2k}}$ *and* $\delta_{j_{2k-1}}$*.*

Recall that $T_{\mathfrak{b}}$ is the set of indices k for which $j_{2k-1} - j_{2k} > 0$; the condition of being relatively even is the same as requiring that $j_{2k-1} - j_{2k} \leq 1$ for all $1 \leq k \leq n$.

Moreover, note that all very even Borel subalgebras satisfy $j_{2k-1} = j_{2k}$, and so are also relatively even.

We then show that all highest weights are compatible for relatively even Borel subalgebras.

Theorem 6.5.16. *A highest weight $\underline{\lambda}_0 \in \Omega_{m|2n}$ is always compatible with any relatively even Borel subalgebra.*

Proof. Let $\mathfrak{b}$ be a relatively even Borel subalgebra and suppose to the contrary that there exists a highest weight incompatible with $\mathfrak{b}$. That is, there exists an index $k \in T_{\mathfrak{b}}$ for which

$$2\lambda_{m-j_{2k}} \leq 2k-1 \leq 2\lambda_{m-j_{2k-1}+1} \leq 2\lambda_{m-j_{2k-1}}.$$

In this case of $j_{2k-1} - j_{2k} = 0$, we have $2k-1 = 2\lambda_{m-j_{2k}} = 2\lambda_{m-j_{2k-1}}$ which forces $2k-1$ to be even, a contradiction.

In the case of $j_{2k-1} - j_{2k} = 1$, we have that $m - j_{2k-1} + 1 = m - j_{2k}$ and thus

$$2\lambda_{m-j_{2k-1}+1} = 2\lambda_{m-j_{2k}},$$

which forces $2k-1 = 2\lambda_{m-j_{2k-1}+1} = 2\lambda_{m-j_{2k-1}+1}$, which forces $2k-1 = 2\lambda_{m-j_{2k-1}+1}$ to be odd, a contradiction. This completes the proof. □

Therefore, by Theorem 6.5.16, if there are no adjacent ϵs in the $\delta\epsilon$ sequence for $\mathfrak{b}$, then all $\mathfrak{b}_{\mathrm{op}}$-highest weights are compatible with $\mathfrak{b}$. With this definition, we can state the following main theorem of this section.

Theorem 6.5.17. *Let $\mathfrak{b} \in \mathcal{B}$. Choose $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}$ and set $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$, where $X_{\mathfrak{b}_e}$ is defined in* (6.5.3). *Define*

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}}.$$

Then for any $\underline{\lambda}_0 \in \Omega_{m|2n}$ that is compatible with $\mathfrak{b}$, we have that

$$SP^*_{\mu,\frac{1}{2}} \circ \tau_{\mathfrak{b}}\left(\underline{\lambda}_{\mathfrak{b}}\right) = SP^*_{\mu,\frac{1}{2}} \circ \tau_0\left(\underline{\lambda}_0\right),$$

where $SP^*_{\mu,\frac{1}{2}}$ is from Remark 4.4.5. Consequently, the eigenvalue of the Capelli operator D^μ on the irreducible component $\left(V^\lambda_{m|2n}\right)^*$ with highest weight $\underline{\lambda}_{\mathfrak{b}}$ is equal to $SP^*_{\mu,\frac{1}{2}}\left(\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})\right)$.

To prove Theorem 6.5.17, we first define a special subset of $T_{\mathfrak{b}}$. Recall that $T_{\mathfrak{b}} = \{1 \leq k \leq n \mid j_{2k} < j_{2k-1}\}$. Then we have the following definition.

Definition 6.5.18. *For each λ that is compatible with $\mathfrak{b}$, define*

$$T_{\mathfrak{b},\lambda} \coloneqq \{k \in T_{\mathfrak{b}} \mid 2\lambda_{m-j_{2k,\mathfrak{b}}} > 2k-1\}.$$

Note that the condition of $T_{\mathfrak{b},\lambda}$ indicates that $\epsilon_{m-j_{2k,\mathfrak{b}}}$ is the term immediately to the right of δ_{2k} in the $\delta\epsilon$-sequence for $\mathfrak{b}$.

Proof of Theorem 6.5.17. It suffices to show that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$. If $\underline{\lambda}_0$ is generic for $\mathfrak{b}$, then this has been shown previously in Theorem 6.5.12.

Suppose $\underline{\lambda}_0$ compatible with, but not generic for, $\mathfrak{b}$. Recall that the set of roots determining $r_{\mathfrak{b}} - r(\underline{\lambda}, \mathfrak{b})$ is given by

$$\begin{aligned}
&\delta_{2\lambda_m+1} - \epsilon_m, \cdots, \delta_{\ell_{m,\mathfrak{b}}} - \epsilon_m, \qquad (**)\\
&\delta_{2\lambda_{m-1}+1} - \epsilon_{m-1}, \cdots, \delta_{\ell_{m-1,\mathfrak{b}}} - \epsilon_{m-1},\\
&\cdots\\
&\delta_{2\lambda_I+1} - \epsilon_I, \cdots, \delta_{\ell_{I,\mathfrak{b}}} - \epsilon_I,
\end{aligned}$$

where I is the least index for which $\ell_{I,\mathfrak{b}} > 2\lambda_I$.

Recall that for each $k \in T_\mathfrak{b}$, $r_{odd,\mathfrak{b},k} = (j_{2k-1} - j_{2k})\delta_{2k-1} - \sum_{i=m-j_{2k-1}+1}^{m-j_{2k}} \epsilon_i$, and that for each index i in this sum, $\ell_i = 2k-1$. The condition of compatibility implies that either

$$\ell_i = 2k-1 < 2\lambda_{m-j_{2k}} \leq 2\lambda_{m-j_{2k}-1} \leq \cdots \leq 2\lambda_{m-j_{2k-1}+1}$$

in which case, $I > i$ and none of these roots appear in the expression (**), or

$$2\lambda_{m-j_{2k}} \leq 2\lambda_{m-j_{2k}-1} \leq \cdots \leq 2\lambda_{m-j_{2k-1}+1} < 2k-1 = \ell_i$$

in which case $I \leq i$ and every root vector that occurs in $r_{odd,\mathfrak{b},k}$ appears in (**).

As in Definition 6.5.18, the set $T_{b,\lambda}$ is the set of $k \in T_\mathfrak{b}$ such that none of the roots determining $r_{odd,\mathfrak{b},k}$ occur among the roots in (**). This set may be empty. It is immediate that

$$r(\underline{\lambda}, \mathfrak{b}) = r(\underline{\lambda}, \mathfrak{b}_e) + \sum_{k \in T_{b,\lambda}} r_{odd,\mathfrak{b},k}.$$

Then we can write

$$\underline{\lambda}_\mathfrak{b} = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}) = \underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}_e) - \sum_{k \in T_{b,\lambda}} r_{odd,\mathfrak{b},k}.$$

Let $M_\mathfrak{b} \in \mathcal{C}_\mathfrak{b}$. Then as $M_\mathfrak{b} \in \mathcal{C}$, we can apply Theorem 6.5.8 to deduce that

$$M_\mathfrak{b}\underline{\lambda}_{\mathfrak{b}_e} + X_{\mathfrak{b}_e} \sim M_0\underline{\lambda}_0 + X_0.$$

Now $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$. Furthermore

$$\begin{aligned} M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} &= M_{\mathfrak{b}}(\underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}_e) - \sum_{k \in T_{b,\lambda}} r_{odd,\mathfrak{b},k}) \\ &= M_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}_e}) - \sum_{k \in T_{b,\lambda}} M_{\mathfrak{b}} r_{odd,\mathfrak{b},k} \\ &= M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}_e} \end{aligned}$$

where the third line uses the definition of $\mathcal{C}_{\mathfrak{b}}$ and follows from the compatibility of $\underline{\lambda}_0$ with $\mathfrak{b}$. □

In the next sections, we address the remaining cases, that is the case when $\underline{\lambda}_0$ is incompatible with $\mathfrak{b}$. We first give a surprising example in $\mathfrak{gl}(2|2)$.

6.6 Surprising example: an incompatible case $\mathfrak{gl}(2|2)$

In this section, let $\mathfrak{g} = \mathfrak{gl}(2|2)$ and $\mathfrak{b} = \delta_2\epsilon_2\epsilon_1\delta_1$. We provide an example in $\mathfrak{gl}(2|2)$ to illustrate that there are cases that we *cannot* express the eigenvalue of $D^{\mu}_{2|2}$ on $\left(V^{\lambda}_{2|2}\right)^*$ as a polynomial function of $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$. We thus would at least like to find a *piecewise* affine map $\tau'_{\mathfrak{b}}$ such that the eigenvalue of $D^{\mu}_{2|2}$ can be expressed as a polynomial function on $\tau'_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$.

Example 6.6.1. Consider the Borel subalgebra $\mathfrak{b}$ defined by the sequence $\delta_2\epsilon_2\epsilon_1\delta_1$. Thus $\ell_1 = \ell_2 = 1$, which is odd, while $j_2 = 0 < j_1 = 2$. Then if $\mathfrak{b}_{\text{op}} = \delta_2\delta_1\epsilon_2\epsilon_1$ we have

$$\mathfrak{b} = r_{\alpha_2}(r_{\alpha_1}(\mathfrak{b}_{\text{op}}))$$

where $\alpha_2 = \delta_1 - \epsilon_1$ and $\alpha_1 = \delta_1 - \epsilon_2$. The possible highest weights are:

$\underline{\lambda}_0$	$\underline{\lambda}_{\mathfrak{b}}$	(condition)
$-(2x, 2y \mid z, z)$	$-(2x-1, 2y-1 \mid z+2, z)$	$x \geq y > 0$
$-(2x, 0 \mid 0, 0)$	$-(2x-1, 0 \mid 1, 0)$	$x > 0$
$(0, 0 \mid 0, 0)$	$(0, 0 \mid 0, 0)$	

We suppose there exists an affine map $y \mapsto M_{\mathfrak{b}} y + X_{\mathfrak{b}}$ satisfying

$$M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} \sim M_0\underline{\lambda}_0 + X_0$$

for all highest weights $\underline{\lambda}_0$, where $\sim$ indicates that they are equivalent under monoidal symmetry.

Set $\underline{\lambda}_0 = -(2x, 2y \mid z, z)$ for some integers $x \geq y \geq 0$ and $z \geq 0$. Note that $M_0\underline{\lambda}_0 = (x, y, z)$. We have from the formulas

$$E_i = \frac{1}{4}(m + 1 - 2n - 2i) \quad \text{and} \quad F_j = \frac{1}{2}(m + 2 + 2n - 4j)$$

with $m = 2, n = 1$ that

$$E_1 = -\frac{1}{4}, \quad E_2 = -\frac{3}{4}, \quad F_1 = 1.$$

Therefore $X_0 = (-\frac{1}{4}, -\frac{3}{4}, 1)$. This yields

$$M_0\underline{\lambda}_0 + X_0 = \begin{bmatrix} \frac{1}{2}x + \frac{3}{4} \\ \frac{1}{2}y + \frac{3}{4} \\ z - 1 \end{bmatrix}.$$

By Proposition 6.5.2, if (6.5.1) holds, then we must have $M_{\mathfrak{b}} \in \mathcal{C}$, so $M_{\mathfrak{b}}$ has the explicit form

$$M_{\mathfrak{b}} = \begin{bmatrix} -\frac{1}{2} & 0 & a & -a \\ 0 & -\frac{1}{2} & b & -b \\ 0 & 0 & c & -1-c \end{bmatrix},$$

for some $a, b, c \in \mathbb{C}$. Moreover, in this case, $r_{\mathfrak{b}} = \alpha_1 + \alpha_2 = 2\delta_1 - \epsilon_1 - \epsilon_2$ so

$$M_{\mathfrak{b}} r_{\mathfrak{b}} = \begin{bmatrix} -\frac{1}{2} & 0 & a & -a \\ 0 & -\frac{1}{2} & b & -b \\ 0 & 0 & c & -1-c \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 2a \\ \frac{1}{2} + 2b \\ 2c \end{bmatrix},$$

and thus $X_{\mathfrak{b}} = (\frac{1}{2} + 2a, \frac{1}{2} + 2b, 2c) + X_0$.

The nongeneric cases impose two conditions on the choices of a, b, c. The first is that $X_{\mathfrak{b}} \sim X_0$ under monoidal symmetry, which implies a strict condition but at least

$$X_{\mathfrak{b}} - X_0 = \begin{bmatrix} \frac{1}{2} + 2a \\ \frac{1}{2} + 2b \\ 2c \end{bmatrix} \in \mathbb{Z}^3. \tag{6.6.1}$$

We now compute $M_{\mathfrak{b}} \underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}}$ in the intermediate nongeneric case, that is, where $x > 0$ but $y = z = 0$. In this case,

$$\underline{\lambda}_{\mathfrak{b}} = \underline{\lambda}_0 - \alpha_2$$

which allows us to rewrite

$$\begin{aligned} M_{\mathfrak{b}} \underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} &= M_{\mathfrak{b}} \underline{\lambda}_0 - M_{\mathfrak{b}} \alpha_2 + M_{\mathfrak{b}}(\alpha_2 + \alpha_1) + X_0 \\ &= (M_{\mathfrak{b}} \underline{\lambda}_0 + X_0) + M_{\mathfrak{b}} \alpha_1 \end{aligned}$$

Since $M_{\mathfrak{b}}\underline{\lambda}_0 + X_0 = M_0\underline{\lambda}_0 + X_0$ for any choice of $M_{\mathfrak{b}}, M_0$, we infer that (6.5.1) is in this case tantamount to $M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} \sim (M_{\mathfrak{b}}\underline{\lambda}_0 + X_0) + M_{\mathfrak{b}}\alpha_1$. We compute

$$M_{\mathfrak{b}}\alpha_1 = \begin{bmatrix} -\frac{1}{2} & 0 & a & -a \\ 0 & -\frac{1}{2} & b & -b \\ 0 & 0 & c & -1-c \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ \frac{1}{2}+b \\ c \end{bmatrix}$$

Again, we infer that if monoidal symmetry holds, then necessarily $M_{\mathfrak{b}}\alpha_1 \in \mathbb{Z}^3$, that is

$$\begin{bmatrix} a \\ \frac{1}{2}+b \\ c \end{bmatrix} \in \mathbb{Z}^3. \tag{6.6.2}$$

Now note that (6.6.1) and (6.6.2) are contradictory: for example, if $a \in \mathbb{Z}$ then $\frac{1}{2} + 2a \notin \mathbb{Z}$. ♠

This is a surprising result. We have found that there is a line in the span of the $\mathfrak{b}$-highest weights on which the interpolation of the affine function valid on all other highest weights does not give a result compatible with the monoidal symmetry of the interpolation super Jack polynomial.

Proposition 6.6.2. *In Example 6.6.1, we conclude that $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$ is not given by the same formula $M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}}$ on all weights $\underline{\lambda}_{\mathfrak{b}}$.*

However, if we define

$$\mathcal{C}_{\mathfrak{b}} = \{M \in \mathcal{C} \mid M(\alpha_1 + \alpha_2) = 0\}$$

and

$$\mathcal{C}_{\mathfrak{b},inc} = \{M \in \mathcal{C} \mid M\alpha_1 = 0\}$$

then we have the following result.

Proposition 6.6.3. *For* $\mathfrak{b}$ *as above, and for any* $M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}$ *and* $M_{inc} \in \mathcal{C}_{\mathfrak{b},inc}$*, we define* $X_{\mathfrak{b}} = X_0$*. Then the formula*

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = \begin{cases} M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} & \text{if } \underline{\lambda}_0 \text{ is generic for } \mathfrak{b}, \\ M_{inc}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} & \text{if } \underline{\lambda}_0 \text{ is not generic for } \mathfrak{b} \end{cases}$$

satisfies $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$ *for all* $\underline{\lambda}_0 \in \Omega_{2|2}$*. Therefore, we conclude that* $SP^*_{\mu,\frac{1}{2}}\left(\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})\right) = SP^*_{\mu,\frac{1}{2}}\left(\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})\right)$.

Proof. We only need to prove the non-generic case. We have that

$$\begin{aligned} M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} &= M_{\mathfrak{b}}\left(\lambda - \alpha\right) + X_0 \\ &= M_{inc}\underline{\lambda}_{\mathfrak{b}} + M_{inc}\alpha + X_0 \\ &= M_0\underline{\lambda}_{\mathfrak{b}} + X_0 = \tau_0(\underline{\lambda}_0) \end{aligned}$$

which completes the proof. □

In the following section, we give a *piecewise-affine function* $\tau_{\mathfrak{b}}$ that satisfies $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$ on all inputs $\underline{\lambda}_0 \in \Omega_{m|2n}$. In general, it will not have the simple form of this example, as the incompatibility condition will not coincide with non-genericity in general.

6.7 Incompatible cases

Recall that we say that a highest weight $\underline{\lambda}_0 \in \Omega_{m|2n}$ (or the corresponding hook Young diagram λ) is *incompatible* with $\mathfrak{b} \in \mathcal{B}$ if there exists $k \in T_{\mathfrak{b}}$ such that

$$2\lambda_{m-j_{2k}} < 2k-1 < 2\lambda_{m-j_{2k-1}+1}.$$

Moreover, we call such k the *incompatible index*. Also recall that ℓ_i is an increasing sequence and $2\lambda_i$ is a decreasing sequence.

Our first goal in this section is to show that ℓ_i and $2\lambda_i$ can cross at most once. That is, if λ is incompatible with $\mathfrak{b}$, then k is unique. We first give an example to illustrate this observation.

Example 6.7.1. Consider the Borel subalgebra $\mathfrak{b} = \epsilon_6\epsilon_5\delta_8\delta_7\delta_6\epsilon_4\epsilon_3\delta_5\delta_4\epsilon_2\epsilon_1\delta_3\delta_2\delta_1$. Then by Example 6.5.10, we have that $T_{\mathfrak{b}} = \{2, 3\}$. Also, we have that

$$(j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8) = (6, 6, 6, 4, 4, 2, 2, 2)\,.$$

Then λ is incompatible if either of the following holds

$$\begin{cases} 2\lambda_2 < 3 < 2\lambda_1 \text{ with } k = 2, \text{ or} \\ 2\lambda_4 < 5 < 2\lambda_3 \text{ with } k = 3. \end{cases}$$

Therefore, the incompatible Young diagrams satisfy

$$2\lambda_1 > 3 > 2\lambda_2 \geq 2\lambda_3 \geq \lambda_4 \geq 2\lambda_5 \geq 2\lambda_6$$

or

$$2\lambda_1 \geq 2\lambda_2 \geq 2\lambda_3 > 5 > \lambda_4 \geq 2\lambda_5 \geq 2\lambda_6.$$

Thus if k exists, it is unique. ♠

Lemma 6.7.2. *Let $\mathfrak{b} \in \mathcal{B}$. Let $\underline{\lambda}_0 \in \Omega_{m|2n}$ be incompatible with $\mathfrak{b}$. Then there exists a unique $k \in T_{\mathfrak{b}}$ such that $2\lambda_{m-j_{2k}} < 2k-1 < 2\lambda_{m-j_{2k-1}+1}$.*

Proof. The existence follows from the definition of $\underline{\lambda}_0$ being incompatible with $\mathfrak{b}$. To show uniqueness, we first consider an adjacent ϵ sequence to the left of $\epsilon_{m-j_{2k}} \cdots \epsilon_{m-j_{2k-1}+1}$. That is, the sequence has the form of

$$\delta_{2p}\epsilon_{m-j_{2p}}\epsilon_{m-j_{2p}-1} \ldots \epsilon_{m-j_{2p-1}+1}\delta_{2p-1}$$

for some $p > k$. Then as $p > k$ and $\epsilon_{m-j_{2p-1}+1}$ is to the left of $\epsilon_{m-j_{2k}}$, we have

$$2\lambda_{m-j_{2p-1}+1} \leq 2\lambda_{m-j_{2k}} < 2k-1 < 2p-1.$$

Thus we may conclude that p is not an incompatible index. Similarly, consider an adjacent ϵ sequence to the right of $\epsilon_{m-j_{2k}} \cdots \epsilon_{m-j_{2k-1}+1}$. That is, the sequence has the form of $\delta_{2q}\epsilon_{m-j_{2q}}\epsilon_{m-j_{2q}-1} \ldots \epsilon_{m-j_{2q-1}+1}\delta_{2q-1}$ for some $q < k$. An similar argument yields

$$2\lambda_{m-j_{2q}} \geq 2\lambda_{m-j_{2p-1}+1} > 2k-1 > 2p-1.$$

Therefore, q is not an incompatible index, and in turn, we conclude that if $\underline{\lambda}_0$ is incompatible with k being an incompatible index, then k is unique. □

Recall the index $I = I_{\lambda,\mathfrak{b}}$ defined in Definition 6.4.12 is the least index for which

$\ell_{I,\mathfrak{b}} > 2\lambda_I$. Next we connect I with the incompatible index. Consider the sequence

$$\cdots \delta_{2k}\epsilon_{m-j_{2k}}\epsilon_{m-j_{2k}-1}\cdots\epsilon_{m-j_{2k-1}+1}\delta_{2k-1}\cdots .$$

such that k is an incompatible index. Then there exists $J \in [m - j_{2k-1}+1, m-j_{2k})$ such that $2\lambda_{J+1} < 2k-1 < 2\lambda_J$. Then we have the following lemma.

Lemma 6.7.3. *Let J be defined above. Then $I_{\lambda,\mathfrak{b}} = J+1$.*

Proof. First notice that

$$\ell_{m-j_{2k}} = \cdots = \ell_{m-j_{2k}-1} = 2k-1. \tag{6.7.1}$$

Then by (6.7.1), we have $2\lambda_{J+1} < 2k-1 = \ell_{J+1,\mathfrak{b}}$. Thus $I \le J+1$ by the definition of I. Now suppose that $I \le J$. Then by the inequalities $\ell_1 \le \cdots \le \ell_m$, it follows that $\ell_{I,\mathfrak{b}} \le \ell_{J,\mathfrak{b}}$. Moveover, $2\lambda_J > 2k-1$ and $\ell_{J,\mathfrak{b}} = 2k-1$ implies $2\lambda_I \ge 2\lambda_J > \ell_{J,\mathfrak{b}} \ge \ell_{I,\mathfrak{b}}$, which contradicts the definition of I. Thus we conclude $I = J+1$. □

Definition 6.7.4. *We say p is in an* incompatible range *if there exists k such that $m - j_{2k-1} + 1 < p \le m - j_{2k}$.*

Lemma 6.7.5. *Let $\mathfrak{b} \in \mathcal{B}$. If $I_{\lambda,\mathfrak{b}}$ is in the incompatible range, then λ is incompatible with $\mathfrak{b}$.*

Proof. Suppose that $I_{\lambda,\mathfrak{b}}$ is in an incompatible range. Then there exists k such that

$$m - j_{2k-1} + 1 \le I_{\lambda,\mathfrak{b}} - 1 < I_{\lambda,\mathfrak{b}} \le m - j_{2k}.$$

Then we have that

$$2\lambda_{m-j_{2k}} \leq 2\lambda_{I_{\lambda,\mathfrak{b}}} \leq 2\lambda_{I_{\lambda,\mathfrak{b}}-1} \leq 2\lambda_{m-j_{2k-1}+1}$$

and

$$\ell_{m-j_{2k}} = \ell_{I_{\lambda,\mathfrak{b}}} = \ell_{m-j_{2k-1}} = 2k-1.$$

Recall that $I_{\lambda,\mathfrak{b}}$ is the least index so that $2\lambda_{I_{\lambda,\mathfrak{b}},\mathfrak{b}} < \ell_{I_{\lambda,\mathfrak{b}},\mathfrak{b}} = 2k-1$. Thus we have $2\lambda_{I_{\lambda,\mathfrak{b}}-1} > \ell_{I_{\lambda,\mathfrak{b}}-1} = 2k-1$. Therefore, we have that

$$2\lambda_{m-j_{2k}} \leq 2\lambda_{I_{\lambda,\mathfrak{b}}} < 2k-1 < 2\lambda_{I_{\lambda,\mathfrak{b}}-1} \leq 2\lambda_{m-j_{2k-1}+1}.$$

Thus, λ is incompatible with $\mathfrak{b}$. □

This lemma implies that if, for given pair $(\mathfrak{b}, \lambda)$, we can find $I_{\lambda,\mathfrak{b}}$ such that $\ell_{I_{\lambda,\mathfrak{b}}} = \ell_{I_{\lambda,\mathfrak{b}}-1}$, then λ is incompatible with $\mathfrak{b}$. Now let us deduce how we must choose our affine map in the case of incompatibility. Recall that we have

$$r_{odd,\mathfrak{b},k} = \left(\delta_{2k-1} - \epsilon_{m-j_{2k}}\right) + \cdots + \left(\delta_{2k-1} - \epsilon_{I_{\lambda,\mathfrak{b}}}\right) \tag{6.7.2}$$

$$+ \left(\delta_{2k-1} - \epsilon_{I_{\lambda,\mathfrak{b}}-1}\right) + \cdots + \left(\delta_{2k-1} - \epsilon_{m-j_{2k-1}+1}\right) \tag{6.7.3}$$

Recall from the proof of Theorem 6.5.17 that (**) is the set of roots we are *not* subtracting. Thus in the incompatible case, we may redefine our $r(\underline{\lambda}, \mathfrak{b})$ as follows. Let $R(\mathfrak{b}, I_{\lambda,\mathfrak{b}})$ be the sum of roots in (6.7.3). That, is

$$R(\underline{\lambda}, \mathfrak{b}, I_{\lambda,\mathfrak{b}}) = \left(I_{\lambda,\mathfrak{b}} - m + j_{2k}) - 1\right)\delta_{2k-1} + \sum_{i=m-j_{2k-1}+1}^{I_{\lambda,\mathfrak{b}}-1} \epsilon_i.$$

Then we have

$$r(\underline{\lambda}, \mathfrak{b}) = r(\underline{\lambda}, \mathfrak{b}_e) + R(\underline{\lambda}, \mathfrak{b}, I_{\lambda,\mathfrak{b}}) + \sum_{k \in T_{\mathfrak{b},\lambda}} r_{odd,\mathfrak{b},k}.$$

Note that if k is in $T_{\mathfrak{b},\lambda}$, then $2k-1 \neq \ell_{I_{\lambda,\mathfrak{b}}}$. Our next goal is to define a subset of $\mathcal{C}$ that vanishes on the additional terms $R(\underline{\lambda}, \mathfrak{b}, I_{\lambda,\mathfrak{b}})$ and $\sum_{k \in T_{\mathfrak{b},\lambda}} r_{odd,\mathfrak{b},k}$.

Definition 6.7.6. *For any index I such that $\ell_I = 2k-1$ for some $1 \leq k \leq n$, define*

$$\mathcal{C}_{\mathfrak{b},I} = \{M \in \mathcal{C} \mid MR(\underline{\lambda}, \mathfrak{b}, I) = 0, Mr_{odd,\mathfrak{b},k} = 0 \text{ for } 2k-1 \neq \ell_I\}.$$

We now can state our main theorem of this section.

Theorem 6.7.7. *Let $\mathfrak{b}$ be an arbitrary Borel subalgebra of $\mathfrak{g}$ with decreasing $\delta\epsilon$ sequence. Set $X_{\mathfrak{b}} = X_{\mathfrak{b}_e}$. Define*

$$\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = \begin{cases} M_{\mathfrak{b}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} & \text{for any } M_{\mathfrak{b}} \in \mathcal{C}_{\mathfrak{b}}, \text{ if } \underline{\lambda}_0 \text{ is compatible with } \mathfrak{b} \\ M_{b,I_{\lambda,\mathfrak{b}}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} & \text{for any } M_{b,I_{\lambda,\mathfrak{b}}} \in \mathcal{C}_{\mathfrak{b},I_{\lambda,\mathfrak{b}}}, \text{ if } \underline{\lambda}_0 \text{ is incompatible with } \mathfrak{b}. \end{cases}$$

Then for every $\underline{\lambda}_0 \in \Omega_{m|2n}$, we have that

$$SP^*_{\mu,\frac{1}{2}} \circ \tau_{\mathfrak{b}}\left(\underline{\lambda}_{\mathfrak{b}}\right) = SP^*_{\mu,\frac{1}{2}} \circ \tau_0\left(\underline{\lambda}_0\right),$$

where $SP^*_{\mu,\frac{1}{2}}$ is from Remark 4.4.5. Consequently, the eigenvalue of the Capelli operator $D^{\mu}_{m|2n}$ on the irreducible component $\left(V^{\lambda}_{m|2n}\right)^*$ with highest weight $\underline{\lambda}_{\mathfrak{b}}$ is equal to $SP^*_{\mu,\frac{1}{2}}\left(\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})\right)$.

Proof. The case where $\underline{\lambda}_0$ is compatible with $\mathfrak{b}$ has been proved in Theorem 6.5.17.

We only need to prove the incompatible case. We have that

$$\begin{aligned}
M_{b,I_{\lambda,\mathfrak{b}}}\underline{\lambda}_{\mathfrak{b}} + X_{\mathfrak{b}} &= M_{b,I_{\lambda,\mathfrak{b}}}\left(\underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b})\right) + X_{\mathfrak{b}} \\
&= M_{b,I_{\lambda,\mathfrak{b}}}\left(\underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}_e) - R(\underline{\lambda}, \mathfrak{b}, I_{\lambda,\mathfrak{b}}) - \sum_{k\in T_{\mathfrak{b},\lambda}} r_{odd,\mathfrak{b},k}\right) + X_{\mathfrak{b}} \\
&= M_{b,I_{\lambda,\mathfrak{b}}}\left(\underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}_e)\right) + X_{\mathfrak{b}},
\end{aligned}$$

by the definition of $\mathcal{C}_{\mathfrak{b},I_{\lambda,\mathfrak{b}}}$. Since $\mathcal{C}_{\mathfrak{b},I_{\lambda,\mathfrak{b}}}$ is a subset of $\mathcal{C}$ and $\mathfrak{b}_e$ is very even, the term $M_{b,I_{\lambda,\mathfrak{b}}}\left(\underline{\lambda}_0 - r(\underline{\lambda}, \mathfrak{b}_e)\right) + X_{\mathfrak{b}}$ is equivalent to $M_{b,I_{\lambda,\mathfrak{b}}}\underline{\lambda}_0 + X_0$ by Proposition 6.5.6, which completes the proof. □

This theorem completes the story of this chapter. By Theorem 6.5.12, Theorem 6.5.17 and Theorem 6.7.7, we conclude that a solution to the refined CEP for $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$ can be found by defining affine transformations or piecewise affine transformations $\tau_{\mathfrak{b}}$ such that $SP^*_\mu \circ \tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) = SP^*_\mu \circ \tau_0(\underline{\lambda}_0)$.

Appendix A

A detailed calculation for $\mathfrak{gl}(1|2n)$

In this section, let $\mathfrak{g} = \mathfrak{gl}(1|2n)$. We outline a detailed computation and a refined solution to the CEP of $\left(\mathfrak{gl}(1|2n), \mathcal{S}^2\left(\mathbb{C}^{1|2n}\right)\right)$ in order to give the readers an idea of how we derive a refined solution to the CEP of $\left(\mathfrak{gl}(m|2n), \mathcal{S}^2\left(\mathbb{C}^{m|2n}\right)\right)$. Recall that the decreasing $\delta\epsilon$-sequence associated to the opposite standard Borel subalgebra of $\mathfrak{gl}(1|2n)$ has the form of

$$\delta_{2n} \ldots \delta_1 \epsilon_1.$$

From Equation (6.1.5), recall that we have

$$E_1 + \frac{1}{2}F_j = \frac{3}{4} - j.$$

Consider the Borel subalgebra $\mathfrak{b}$ whose corresponding $\delta\epsilon$-sequence is $\ldots \delta_{2j+1}\epsilon_1\delta_{2j}\delta_{2j-1}\ldots$. With respect to $\mathfrak{b}$, the possible highest weights of $\left(V^{\lambda}_{1|2n}\right)^*$ are

$$\underline{\lambda}_{\mathfrak{b}} = -\left(2\lambda_1 - 2i \mid \mu_1+1, \mu_1+1, \ldots, \mu_i+1, \mu_i+1, \mu_{i+1}, \mu_{i+1}, \ldots, \mu_n, \mu_n\right) \text{ for } 0 \leq i \leq j$$

such that

$$2\lambda_1 - 2i + \mu_{j-k} = 0 \text{ for all } 0 \le k \le j-i-1.$$

We claim that the affine map $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$ given by $M\underline{\lambda}_{\mathfrak{b}} + X$, where

$$M = \begin{pmatrix}
-\frac{1}{2} & j-\frac{1}{2} & \frac{1}{2}-j & j-\frac{3}{2} & \frac{3}{2}-j & j-\frac{5}{2} & \frac{5}{2}-j & \cdots & \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -1\cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

and

$$X = \begin{pmatrix} E_1 + j \\ F_1 - 1 \\ \vdots \\ F_j - 1 \\ F_{j+1} \\ \vdots \\ F_n \end{pmatrix},$$

satisfies $\tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}}) \sim \tau_0(\underline{\lambda}_0)$. Notice that M is an example of Definition 6.2.1. Then we have that

$$M\underline{\lambda}_{\mathfrak{b}} + X = \begin{pmatrix} \lambda_1 - i + E_1 + j \\ \mu_1 + F_1 \\ \vdots \\ \mu_i + F_i \\ \mu_{i+1} + F_{i+1} - 1 \\ \vdots \\ \mu_j + F_j - 1 \\ \mu_{j+1} + F_{j+1} \\ \vdots \\ \mu_n + F_n \end{pmatrix}$$

which is equal to

$$\begin{pmatrix} \lambda_1 + E_1 + j - i \\ \mu_1 + F_1 \\ \vdots \\ \mu_i + F_i \\ \mu_{j-(j-(i+1))} + F_{j-(j-(i+1))} - 1 \\ \vdots \\ \mu_{j-k} + F_{j-k} - 1 \\ \vdots \\ \mu_{j-0} + F_{j-0} - 1 \\ \mu_{j+1} + F_{j+1} \\ \vdots \\ \mu_n + F_n \end{pmatrix}. \tag{A.0.1}$$

Our claim is that for each $0 \leq i \leq j$, we can use the monoidal symmetry property to show that $M\underline{\lambda}_{\mathfrak{b}} + X$ is equivalent to

$$\begin{pmatrix} \lambda_1 + E_1 + j - i - (k+1) \\ \mu_1 + F_1 \\ \vdots \\ \mu_i + F_i \\ \mu_{j-(j-(i+1))} + F_{j-(j-(i+1))} - 1 \\ \vdots \\ \mu_{j-(k+1)} + F_{j-(k+1)} - 1 \\ \mu_{j-k} + F_{j-k} \\ \vdots \\ \mu_{j-0} + F_{j-0} \\ \mu_{j+1} + F_{j+1} \\ \vdots \\ \mu_n + F_n \end{pmatrix} \tag{A.0.2}$$

for $0 \leq k \leq j - i - 1$. We prove this by using induction on k. When $k = 0$, we have that $2\lambda_1 - 2i + \mu_j = 0$ and in particular, by considering the first row and the μ_j-row of the vector in (A.0.1), we have

$$\begin{aligned} &(\lambda_1 + E_1 + j - i) + \frac{1}{2}(\mu_j + F_j - 1) \\ &= \left(\lambda_1 + \frac{1}{2}\mu_j\right) + \left(E_1 + \frac{1}{2}F_j\right) + j - i - \frac{1}{2} \\ &= i + \frac{3}{4} - j + j - i - \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Thus by using the monoidal symmetry property in Corollary 6.1.4, the vector (A.0.1) is equivalent to

$$\begin{pmatrix} \lambda_1 + E_1 + j - i - 1 \\ \mu_1 + F_1 \\ \vdots \\ \mu_i + F_i \\ \mu_{j-(j-(i+1))} + F_{j-(j-(i+1))} - 1 \\ \vdots \\ \mu_{j-k} + F_{j-k} - 1 \\ \vdots \\ \mu_{j-1} + F_{j-1} - 1 \\ \mu_{j-0} + F_{j-0} \\ \mu_{j+1} + F_{j+1} \\ \vdots \\ \mu_n + F_n \end{pmatrix} \tag{A.0.3}$$

which is the vector (A.0.2) when $k = 0$. Now suppose that the claim is true for $k \geq 0$, that is, $M\underline{\lambda}_{\mathfrak{b}} + X$ has been transformed into vector (A.0.2). We show this is claim also holds for $k+1$. Now consider the first row, and $\mu_{j-(k+1)}$-row of vector (A.0.2). We have the following computation

$$\begin{aligned}
&(\lambda_1 + E_1 + j - i - (k+1)) + \frac{1}{2}\left(\mu_{j-(k+1)} + F_{j-(k+1)} - 1\right) \\
&= \left(\lambda_1 + \frac{1}{2}\mu_{j-(k+1)}\right) + \left(E_1 + \frac{1}{2}F_{j-(k+1)}\right) + j - i - (k+1) - \frac{1}{2} \\
&= i + \frac{3}{4} - (j - (k+1)) + j - i - (k+1) - \frac{1}{2} \\
&= i + \frac{3}{4} - j + (k+1) + j - i - (k+1) - \frac{1}{2}
\end{aligned}$$

$$= \frac{1}{4}.$$

Thus the vectors in (A.0.1) and (A.0.3) are equivalent. Thus, when $k = j - i - 1$, we have that $M\underline{\lambda}_{\mathfrak{b}} + X \sim \tau_0(\underline{\lambda_0})$. Therefore since $SP^*_\mu \in \Lambda_{1,n,\frac{1}{2}}$, and $SP^*_\mu \circ \tau_0(\underline{\lambda_0})$ is a solution to the Capelli Eigenvalue Problem of $\left(\mathfrak{gl}(1|2n), \mathcal{S}^2\left(\mathbb{C}^{1|2n}\right)\right)$, it follows that $SP^*_\mu \circ \tau_{\mathfrak{b}}(\underline{\lambda}_{\mathfrak{b}})$ is a solution to the Capelli Eigenvalue Problem of $\left(\mathfrak{gl}(1|2n), \mathcal{S}^2\left(\mathbb{C}^{1|2n}\right)\right)$

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